Solving DSGE Models with Incomplete Markets: Global vs. Local Methods*

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Abstract

This paper revisits the implications of solving small open economy models with incomplete asset markets using stationarity inducing techniques. In doing so, we compare model-based solutions with ad-hoc perturbation methods, in particular, global solutions with Bewley-Aiyagari-Huggett (BAH) or Uzawa-Epstein (UE) preferences vs. log-linear, second-order approximation, and risk-adjusted steady state solutions with a debt-elastic interest rate function. Our analysis in the time and frequency domain highlights two major challenges for perturbation methods. First, for perturbation methods to capture the dynamics implied by nonlinear methods, the modeler would need to know the true solution of the global model. Second, even if the modeler has the true solution in hand, the (optimally parameterized) perturbation-based solutions are not able to generate the desired cyclical properties at all frequencies.

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1 Introduction

The open-economy macroeconomics literature has increasingly relied on dynamic stochastic general equilibrium (DSGE) models to shed light on important research questions. In doing so, researchers have had to take a stance on what induces stationarity of net foreign asset (NFA) holdings in their models.

Schmitt-Grohé and Uribe (2003) surveys the commonly-used stationarity-inducing techniques and analyze how these techniques differ from each other in accounting for macroeconomic fluctuations. In their analysis, they focus on endogenous discounting (as in Mendoza (1991), among others), a debt-elastic interest rate function (as in Schmitt-Grohé and Uribe (2001)), and convex-portfolio adjustment costs (as in Neumeyer and Perri (2005)). The authors found that, using a log-linear approximation of the model around a common deterministic steady-state, all stationarity inducing techniques yielded nearly identical results at the business cycle frequency. Relying on these findings, a large body of the literature made use of these ad-hoc techniques to shed light on important issues not only related to the movements at the business cycle frequency but also at lower frequency (such issues surrounding international portfolio choice problems, capital flows, global imbalances, as well as for studying rare-events, such as financial crises).

In this paper, we revisit the role of stationarity inducing techniques from a broader perspective. In particular, we compare the implications of the debt-elastic interest rate function–the most commonly used ad-hoc stationarity inducing technique–(DEIR, henceforth) solved with perturbation methods to model-based solutions with endogenous discounting (Uzawa-Epstein type preferences, UE, henceforth) and Bewley-Aigari-Huggett type preferences (BAH, henceforth) solved with global methods. We compare these solution methods both in a simple endowment economy model as well as in a standard small open economy real business cycle (RBC) model. In solving the DEIR model with perturbation methods, we use log-linear and second-order approximation techniques around a deterministic steady state as well considering a first-order approximation around a risk-adjusted steady state (in the spirit of Coeurdecier, Rey and Winant (2011)). We explore the low frequency dynamics (such as the behavior of the unconditional mean NFA positions under various assumptions on exogenous disturbances), long-run moments and the frequency domain decomposition of important aggregate variables. We find that the dynamics of the perturbation-based solutions can differ significantly from those of the model-based solutions at both low and business cycle frequencies.

For example, when we adjust the variability or persistence of the exogenous endowment process, the perturbation-based methods yield significantly different behavior of the NFA positions than the model-based solutions. The model-based solutions imply an increase in the NFA position (with a stronger response in the BAH model relative to the UE model) to more variability in the endowment process, capturing the precautionary saving motive of agents. Naturally, a log-linear approximation strips away any precautionary savings behavior from the model. The second-order approximation can be parameterized (by searching for an appropriate value of the debt elasticity of the interest rate and an appropriate deterministic steady state value of the NFA position) to closely match the precautionary savings behavior in the BAH setup when endowment variability is between zero and its baseline value. However, this parameterization induces the second-order approximation to significantly overstate the precautionary savings motive in BAH as the variability of the endowment process increases above its baseline value. Moreover, in this exercise, relationship between the autocorrelation of the NFA position and the autocorrelation of the net export-to-GDP ratio (NX) differs significantly from the relationship implied by the model-based solutions.

Further discrepancies exist and are discussed in detail. More generally, our analysis shows that finding the right combination of parameter values that would imply an appropriate behavior of NFA positions can be a severe challenge for modelers. In order to match the behavior implied by the model-based solutions, the modeler would need to have the true nonlinear solution in hand. Otherwise, it would practically be impossible for the modeler to pin down the right combination of parameter values. In our exercises, we find that the behavior of the NFA, and by extension NX are very sensitive to small change in the elasticity of the interest rate function.

Our findings with the RBC cycle model echoes those results from the endowment economy model. In addition, we find that the perturbation-based methods imply excess variability in several macroeconomic aggregates (e.g., consumption, investment, and net exports). Moreover, the discrepancy in the perturbation-based methods relative to the model-based solutions become more pronounced when the underlying shock process becomes more variable or more persistent.

Our paper is related to a large literature on open-economy DSGE models. On emerging market business cycles, several papers (Aguiar and Gopinath (2006), Boz, Daude and Durdu (2011), Neumeyer and Perri (2005), Mendoza (1991, 1995, among others), Uribe and Yue (2006)) have used variants of small open economy business cycle models to shed light on important emerging market regularities. The papers that relied on ad-hoc solution methods justified their assumption based on the results of Schmitt-Grohé and Uribe (2003). In our broader analysis, the naive reliance on ad-hoc perturbation-based methods can be shown to generate misleading inference about equilibrium dynamics of important macroeconomic variables.

The rest of the paper is organized as follows. Section 2 presents the differences implied by the solution methods in a simple endowment economy model. Section 3 provides a similar analysis in a standard small open economy RBC model. Section 4 summarizes our conclusions and offers suggestions for further research.

2 Endowment economy model

2.1 Structure of the model

Consider a small open economy inhabited by a representative agent, whose aggregate consumption is denoted by c. The agent's preferences are given by

$$U = E_0 \left\{ \sum_{t=0}^{\infty} \exp\left[-\sum_{\tau=0}^{t-1} \nu(c_\tau) \right] u(c_t) \right\}. \tag{1}$$

Functional forms are given by

$$\nu(c) = \rho^{UE} \ln(1+c) \;\text{ or }\; \ln\left(1+\rho^{BAH}\right),$$

$$u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}.$$

Period utility has the form of constant relative risk aversion (CRRA), and σ is the relative risk aversion coefficient. The time preference function $\nu(c)$ takes one of two forms. With the UE formulation, the rate of time preference is endogenous and given by $v(c) = \rho^{UE}\ln(1+c)$, where $\rho^{UE} > 0$ measures the elasticity of the rate of time preference with respect to $1+c$. With the BAH formulation, the rate of time preference is given by the standard constant fraction $0 < \rho^{BAH} < 1$ (i.e., the typical exogenous discount factor is $\beta \equiv 1/(1+\rho^{BAH})$).

The economy chooses consumption and foreign assets to maximize Eq. (1) subject to the standard resource constraint

$$c_t = e^{z_t} y + b_t - q_t b_{t+1}, \tag{2}$$

where b denotes the level of foreign assets that the agent accumulates through one-period discount bonds traded in a frictionless global credit market. These bonds pay one unit of consumption goods in the following period. Denoting the world interest rate r_t, the price of the bond is simply equal to

$q_t \equiv \frac{1}{1+r_t}$. The economy's mean or trend income, y, is subject to random shocks, z_t of the following form

$$z_t = \rho_z z_{t-1} + \sigma_z \varepsilon_t^z,$$

where ε_t^z are i.i.d. standard normal innovations.

2.2 Equilibrium

The optimization problem of this small open economy is analogous to the optimization problem of a single individual in the heterogenous-agent models of precautionary savings (e.g., Aiyagari (1994) and Hugget (1993)). As in those models, CRRA utility implies that the marginal utility of consumption goes to infinity as consumption goes to zero from above, making the economy extremely averse to consumption and savings plans that would leave it exposed to the risk of very low consumption at any date and state of nature. To rule out these plans, agents in this economy impose on themselves Aiyagari's Natural Debt Limit, by which they never borrow more than the annuity value of the worst realization of income $b_{t+1} \geq -\min(e^{z_t} y)/r$. In addition, following Aiyagari (1994), we can impose an ad-hoc debt limit ϕ such that $b_{t+1} \geq \phi \geq -\min(e^{z_t} y)/r$.

If the ad-hoc debt limit does not bind, the optimality condition of the economy's maximization problem is

$$q_t U_c(t) = \exp\left(-\nu(c_t)\right) E_t \left[U_c(t+1)\right], \tag{3}$$

where $U_c(t)$ denotes the lifetime marginal utility of date-t consumption. In the BAH setup, $U_c(t)$ is just the standard period marginal utility of c_t. In the UE setup, however, $U_c(t)$ includes both the period marginal utility of c_t and the impatience effects by which changes in c_t affect the subjective discounting of all future utility flows after date t.

A competitive equilibrium for this economy is defined by stochastic sequences $[c_t, b_{t+1}]_{t=0}^{\infty}$ that satisfy the Euler Eq. (3) and the resource constraint, Eq. (2) for all t. In this economy, if agents have access to complete insurance markets to fully diversify away all the risk of endowment fluctuations, the equilibrium would feature a constant consumption stream and the economy's wealth position would be time- and state-invariant. If the asset market is incomplete, as is the case in our setup, the wealth position changes over time and across states of nature and consumption cannot attain a perfectly smooth path.

In addition to the model-based approaches with BAH or UE preferences described so far, modelers have employed several other ad-hoc methods to characterize a stationary equilibrium. It has increasingly become common practice to use perturbation methods in which stationarity is induced through a DEIR function of the following form

$$\frac{1}{q_t} = 1 + r_t = 1 + r + \psi \left[e^{b_{t+1} - b} - 1 \right], \tag{4}$$

where b is the level of foreign assets in the deterministic steady state and ψ is the debt elasticity of the interest rate. With this formulation, by assumption, the interest rate at the deterministic steady state is equal to the rate of time preference, $\beta = \frac{1}{1+r}$. With the BAH formulation, the economy attains a well-defined long-run distribution of foreign assets with $\beta[1 + r] < 1$.[1] With the UE formulation, a well-defined long-run distribution of assets exists with $\rho^{UE} \leq \sigma$ (see Epstein, 1983).[2]

The competitive equilibrium can be characterized in recursive form that solves the following Bellman equation for the BAH case

$$V(b_t, z_t) = \max \left\{ \frac{c_t^{1-\sigma}}{1-\sigma} + \beta E_t V(b_{t+1}, z_{t+1}) \right\}, \tag{5}$$

subject to

$$c_t = e^{z_t} y - b_t + q_t b_{t+1}.$$

With UE, the endogenous discount factor, $\beta(c) \equiv (1 + c)^{-\rho^{UE}}$, will replace the constant discount factor β. For both the BAH and UE setups, we solve for the equilibrium dynamics using a value function iteration on the Bellman Eq. (5). With perturbation methods, however, we derive the log-linear and second-order approximation of the optimality conditions around their deterministic steady state values.[3]

The risk-adjusted steady solution method follows Coeurdacier et al. (2011) in which a second-order approximation of the conditional expectation of the steady state equations is found and around which a first-order approximation of the model's dynamics is taken.[4] In other words,

[1]In this case, the long-run averages of assets and consumption are finite. In contrast, with $\beta(1 + r) \geq 1$, assets diverge to infinity in the long run because marginal utility converges to zero almost surely (see Ch. 17 in Ljungqvist and Sargent, 2004).

[2]Foreign assets converge to a well-defined long-run distribution because the rate of time preference increases (decreases) relative to the interest rate if consumption and assets rise (fall) too much in the long run, and this changes incentives for savings in favor of reducing (increasing) asset holdings.

[3]Our implementation of the perturbation methods closely follows Schmitt-Grohé and Uribe (2004).

[4]Our implementation of the risk-adjusted steady state varies slightly from Coeurdacier et al. (2011). In particular, Coeurdacier et al. (2011) not only take a conditional second order approximation of equilibrium equations for the steady state but also take a conditional second-order approximation of the equilibrium equations' Jacobian, effectively requiring a third-order approximation of the equilibrium equations. This second step is crucial to induce stationarity

the risk-adjusted steady state around which the model is approximated takes into account future risk and the steady state net foreign asset position (and capital in the real business cycle model) should, in theory, capture agent's precautionary savings motive. Details of the solution method can be found in the Appendix.

In a series of quantitative exercises, we highlight the main differences implied by the use of the perturbation-based methods compared to the model-based solutions as we describe in the next sections.

2.3 Calibration

We calibrate the model so that time periods equal years and so that the deterministic stationary equilibrium using UE preferences matches a set of statistics from the Mexican economy.[5] Table 1 shows the baseline calibration parameters. The CRRA coefficient is set to the standard value $\sigma = 2$. Mean income is normalized to $y = 1$ so that steady-state allocations can be interpreted as ratios relative to average GDP. The steady state real interest rate is set to 5.9%, which is the average of Uribe and Yue's (2006) real interest rate including the EMBI spread for Mexico.

In the UE setup, the value of the time preference elasticity, ρ^{UE}, is calibrated such that unconditional mean of b equals the average of Mexico's net foreign assets-to-GDP ratio, -0.44 based on Lane and Milesi-Ferretti's (2006) estimates over the 1985–2004 period.

The BAH setup does not have a well-defined deterministic stationary equilibrium, because without uncertainty, $\beta(1+r) < 1$ implies that consumption falls at a gross rate of $(\beta(1+r))^{1/\sigma}$ until the economy hits the debt limit. Hence, to complete the calibration of the BAH setup we keep all the parameters as in the UE setup and set ϕ and β to values such that the model with BAH preferences matches the long run average of the NFA position and the cyclical standard deviation of consumption in the Mexican data. In the DEIR setup, β is set to $\frac{1}{1+r}$, and ψ is set to a commonly-used, small value .001. We leave the deterministic steady state value of b, and on occasion ψ as free parameters in our quantitative exercises, in order to give the perturbation-based methods the best chance to match the global solutions. The alternative parameterizations that we consider are discussed in Section 2.4 when we present the quantitative results.

The standard deviation and persistence parameters of the endowment process in the baseline

of the net foreign asset position in the model presented by Coeurdacier et al. (2011), as pointed out by de Groot (2014). However, since we only use the risk-adjusted steady state methodology to solve the debt-elastic interest rate model, which is already stationary, we leave out this second step. The omission of the second step does not, in any significant way, alter the behavior of our solution.

[5]The calibration to Mexico is not critical for our key findings.

calibration is set using values from the Mexican data for the 1965-2005 period. These values are .0327 and .597 for σ_z and ρ_z, respectively. Since there is a great deal of variation in the estimation of these parameters across emerging market economies, we pay particular attention to the ability of the perturbation-based methods to generate predictions consistent with the BAH and UE setups when the endowment process changes. For the model-based solutions, BAH and UE, the income process is estimated using Tauchen and Hussey (1991)'s quadrature algorithm. With perturbation methods, we directly use the autoregressive representation of the income process.

2.4 Quantitative results

Before presenting the results of our quantitative analysis, we examine the relationship between the behavior of the NFA position, b, and net exports, nx. We already know that small open economy models, without ad-hoc additions, generate a unit root in the NFA position. In addition, Garcia-Cicco et al. (2010) find that a small open economy model, estimated to Mexican data, generates a counterfactually flat autocorrelation function of net exports very close to 1, despite the empirical autocorrelation function being clearly downward sloping. Therefore, to understand this relationship between the NFA position and NX, we proceed as follows. First, we postulate the following AR(1) process for b

$$b_{t+1} = (1-\rho)\mu_b + \rho b_t + \varepsilon_{t+1}, \tag{6}$$

where ρ is the autocorrelation of the NFA position and ε_{t+1} is a mean zero stochastic process. This implies that $E[b] = \mu_b$. Next, we introduce the definition of net exports, nx

$$nx_t = q_t b_{t+1} - b_t,$$

where $q = \frac{1}{1+r}$, implying that $E[nx] = (q-1)\mu_b$. In this environment, the relationship between the autocorrelation of net exports, $\rho(nx_t)$, and the autocorrelation of the NFA position is[6]

$$\rho(nx) = \frac{q^2\rho + \rho - q - q\rho^2}{1 + q^2 - 2q\rho}. \tag{7}$$

Figure 1 plots the the theoretical relationship implied by Eq. 7 in the left panel. As that panel highlights, there is a positive and highly nonlinear relationship between the autocorrelations

[6] See the Appendix for further details.

of net exports and the NFA position. As the autocorrelation of NFA position approaches 1, the autocorrelation of net exports becomes increasingly sensitive to small changes in the autocorrelation of the NFA position. This implies that, to capture the true dynamics of net exports, the solution method would need to capture the true dynamics of the NFA position exceedingly accurately. The panel on the right highlights that the simulated model under BAH and UE solutions can deliver this general pattern of nonlinear relationship reasonably well.

We next examine how each solution method affects the behavior of the NFA position and net exports. The top panel of Figure 2 plots the unconditional mean NFA position as a function of the standard deviation of output (on the left) and the autocorrelation of output (on the right). As the left panel indicates, increases in output variability produce large increases in precautionary demand for foreign assets with BAH (solid blue) and UE (dashed green) setups (although the BAH setup always yields higher precautionary savings than the UE setup). The perturbation methods have one free parameter, the deterministic steady state value of b that enters in the debt elastic interest rate function and that we did not calibrate in Section 2.3. We now set this free parameter in order to match the mean NFA position of -0.44 under the baseline calibration of the endowment process for both BAH and UE.[7] In the top panels, the log-linear approximation generates a flat line as it abstracts from any precautionary demand behavior in the model. The risk-adjusted steady state solution yields a behavior similar to the UE setup (dashed pink), displaying only a modest degree of precautionary savings. The second-order approximation under the baseline calibration matches the behavior implied by the BAH setup up to around a 4.5 percent standard deviation of output. With further increases in output variability, the rate of increase of the mean NFA position is decreasing under the BAH setup. The nature of a second-order approximation, however, enforces the relationship between the variability of output and the mean NFA to be convex, causing the accuracy of the approximation to deteriorate along this dimension as the variability of output increases further.

The top right panel shows the results when the autocorrelation of output rises. Overall, the results are qualitatively similar to the experiments shown in the upper left panel. However, while the ad hoc calibration of the the debt elasticity parameter at .001 and the carefully chosen value of b allowed the second-order approximation to capture the mean NFA position across different levels of output variability relatively well, the approximation does much more poorly across different levels

[7]In the first-order approximation, $b = -0.44$, in the risk-adjusted steady state, $b = -0.46$ and in the second-order approximation, $b = -0.57$.

of the persistence of the endowment process, without the freedom to recalibrate those parameters. It is clear that, had we parameterized ψ and b to match the BAH results along the dimension of output persistence, it would have been at the expense of matching the precautionary savings behavior of the BAH setup along the dimension of output variability.

The middle and lower panels show the implications for the autocorrelation of the NFA position and the NX to output ratio, respectively. A key distinction between the BAH and UE setups is that the autocorrelation of NFA positions is virtually insensitive to changes in the variability and persistence of the endowment process in the UE setup, while under BAH, the autocorrelation weakens significantly, both when the variability and the persistence of the endowment process falls. The nonlinearity of the relationship between the autocorrelation of the NFA position and NX shown in Figure 1 explains the corresponding plots for the autocorrelation of NX in the lower panels of Figure 2. While the top panels showed the second-order approximation generating precautionary savings behavior more akin to the BAH set up, in terms of autocorrelations, the second-order approximation now more closely resembles UE. In particular, the autocorrelation function of the second-order approximation is virtually flat as the variability and persistence of the endowment process changes.

The nature of the autocorrelation function of the NFA position and NX in the second-order approximation however, is not a feature of the second-order approximation per se. The two other perturbation methods, namely the log-linear approximation around the deterministic steady state and the first-order approximation around the risk-adjusted steady state, show very similar autocorrelations.

We next examine the implications of the solution methods on the long-run moments, as shown in Table 2. The long-run moments are computed using series generated from random draws for 50 simulations each solved for 4500 periods.[8] Remember that all the solution methods are calibrated to generate nearly identical unconditional mean NFA positions, as the first line shows. Under this calibration, all three perturbation methods lead to significantly more variable NFA positions than the global methods–around 14 percent for the perturbation methods compared with $4\frac{1}{2}$ percent with the BAH method and $7\frac{3}{4}$ percent with the UE method. The higher variability of the NFA position also translates in to the relatively higher variability of NX; around 1.2 percent for perturbation methods, 0.74 percent for BAH and 0.94 percent for UE. Despite the high autocorrelation of the

[8] The model is simulated for 5700 periods with the first 1200 periods dropped. We do not filter the simulated series to avoid potential noise filtering methods might cause.

NFA position, all methods imply stationary and downward slopping behavior for the autocorrelation of net exports.

The long-run moments tell us something about the overall variability of a series but does not tell us whether this variability is the product of high frequency movements in the series or low frequency cycle. To explore this dimension of the small open economy model, we reexamine the simulated series from our various solution methods in the frequency domain. Figure 3 shows the frequency domain representation of the fluctuations of several important aggregate variables. To compute the frequency domain representation, we use nonparametric estimates (or periodograms) of the spectral densities for the simulated series (see Hamilton (1994), chapter 6).[9]

The spectral densities for the log-linear approximation (dotted light blue), second-order approximation (dashed red) appear virtually identical, but they differ significantly from the spectral densities for the BAH and UE setups at all frequencies (the vertical line shows the business cycle frequency). The spectral density for the NFA for the risky-adjusted steady state approximation (dot-dashed indigo) is different from those for the log-linear and the second-order approximation, but for consumption and NX, the spectral densities for the risky-adjusted steady state approximation lines up closely with those of the other perturbation-based methods. Recall that the baseline parametrization of the second-order approximation does a reasonably job in capturing the behavior of the NFA position under the BAH setup. Yet, that solution implies significantly different behavior of cyclical variations at all frequencies. The most noticeable of these discrepancies is the excessive weight placed on low frequency cycles in the perturbation-based methods relative to the global solutions.

These results highlight two major challenges for the perturbation methods. First, for these methods to have their best chance to capture the dynamics implied by nonlinear methods, the modelers would need to know how the true global solution looks like. Without having that information in hand, the modeler would be, in a sense, "shooting in the dark." Second, even if the modeler has the true solution in hand, the perturbation methods, even with the best parameterization, would imply significant differences for the cyclical fluctuations at all frequencies.

The endowment economy model serves the purpose of providing a very clean exposition of the various solution techniques. There is essentially only one intertemporal trade off and no intratem-

[9] The results under the parametric estimates of the spectral densities are similar. To generate the parametric estimates, we first fit an $ARMA(p, q)$ process by identifying the correct lag structure using Bayes-Schwartz (BIC) criterion. Next, we plug the corresponding $ARMA$ coefficients and the variance of the noise process into the formula shown in Eq. 6.1.14 in Hamilton (1994).

poral trade off. All the variation in unconditional means and variation in dynamic behavior is a direct result of consumption smoothing and precautionary savings. The nature question to ask is whether the differences across solution methods that we observe and the difficult that perturbation-based methods have in replicating the implied behavior by global models is simply a product of this stylized environment. In the next section, we show that the results of the endowment economy broadly carry over to the richer RBC economy model with intratemporal tradeoffs and endogenous capital accumulation.

3 Real business cycle model

3.1 Structure of the model

The real business cycle model features production with endogenous capital and labor, and an agent with preferences over labor supply. The model introduces convex costs associated with adjusting capital which are typically introduced in the literature to match the variability and the persistence in investment. The agent can borrow and lend in international capital markets. The asset markets are incomplete, mainly because only financial instrument available is a one-period, non-contingent bond. However, the household is also able to save by accumulating capital. In this framework, the household maximization problem, and the utility function are updated as follows

$$U = E_0 \left\{ \sum_{t=0}^{\infty} \exp\left[-\sum_{\tau=0}^{t-1} \nu(c_\tau, l_\tau) \right] u(c_t, l_t) \right\}. \tag{8}$$

where l is labor. Functional forms are also updated as follows

$$\nu(c,l) = \rho^{UE} \ln\left(1 + c - \frac{l^\omega}{\omega}\right) \text{ or } \ln\left(1+\rho^{BAH}\right),$$

$$u(c_t, l_t) = \frac{\left(c_t - \frac{l_t^\omega}{\omega}\right)^{1-\sigma}}{1-\sigma}.$$

The resource constraint becomes

$$c_t + i_t = \exp(z_t) F(k_t, l_t) - q_t b_{t+1} + b_t - \Phi(k_{t+1}, k_t), \tag{9}$$

where $i_t = k_{t+1} - (1-\delta)k_t$ denotes investment and δ denotes the depreciation rate of capital. The price of the bond, q_t, takes the functional form in Eq. (4), z_t denotes the productivity shock, which takes the same autoregressive process as in the previous section, and the production function $F(k_t, l_t)$ is Cobb-Douglas

$$F(k_t, l_t) = k_t^{\alpha} l_t^{1-\alpha}.$$

Finally, $\Phi(\cdot)$ denotes the convex capital adjustment costs and takes the following functional form

$$\Phi(k_{t+1}, k_t) = (k_{t+1} - k_t)\frac{\kappa}{2}\left(\frac{k_{t+1} - k_t}{k_t}\right).$$

The competitive equilibrium can once again be characterized in recursive form that solves the following Bellman equation for the BAH case

$$V(b_t, k_t, z_t) = \max\left\{\frac{c_t^{1-\sigma}}{1-\sigma} + \beta E_t V(b_{t+1}, k_{t+1}, z_{t+1})\right\},$$

subject to

$$c_t + k_{t+1} = \exp(z_t)F(k_t, l_t) - q_t b_{t+1} + b_t + (1-\delta)k_t - \Phi(k_{t+1}, k_t).$$

As in the endowment economy case, with UE, the endogenous discount factor, $\beta(c) \equiv (1+c)^{-\rho^{UE}}$, replaces the constant discount factor β. As detailed in the solution of the endowment economy model, the BAH and UE setups are solved for the equilibrium dynamics using a value function iteration on the Bellman equation. The perturbation methods are solved using a either log-linear or second-order approximation around the deterministic steady state or first-order approximation around the risk-adjusted steady state. In each case, the constant world interest rate is replaced with the debt elastic interest rate function in order to induce stationarity of the NFA position.

3.2 Calibration

We follow a similar method to calibrate the model so that the deterministic stationary equilibrium using UE preferences matches a set of statistics from the Mexican economy. Table 3 shows the baseline calibration parameters. The CRRA coefficient and the real interest rate are set to the same values as in endowment economy model. The coefficient for the disutility of labor is set to 1.65, and the labor share of output is set to 0.7 following the literature (i.e., Mendoza (1991)). The depreciation rate of capital is set to 0.12 and the investment adjust cost parameter, κ, is set to to match the variability of investment in the Mexican data.

Similar to the endowment economy model, the value of the time preference elasticity in the UE setup, ρ^{UE}, is calibrated such that the unconditional mean NFA position, b, matches the average of Mexico's NFA-to-GDP ratio, -0.44 based on Lane and Milesi-Ferretti (2006) estimates over the 1985–2004 period. In the BAH setup, we set ϕ and β to values such that the model with BAH preferences matches the long-run average of same NFA statistic as well as the cyclical standard deviation of consumption in the Mexican data. In the DEIR setup, β and ψ are kept at the values used in the endowment economy model.

The standard deviation and autocorrelation of the technology process is recalibrated to match the corresponding values for the Mexican GDP data for the 1965-2005 period. Note that the production function introduces amplification to the technology shocks, so that the estimated persistence and the variability of the technology process is lower than those estimated in the endowment economy model. Production requires less persistent and less variable technology shocks than those used in the endowment economy model. Also as in the endowment economy model, for BAH and UE, the finite Markov representation of the technology process is estimated using Tauchen and Hussey (1991)'s quadrature algorithm. With perturbation methods we directly use the autoregressive representation of the technology process.

3.3 Quantitative results

Table 4 presents the unfiltered, unconditional moments of our five solution methods. The first noticeable difference between the global and perturbation methods is that the perturbation methods are not able to capture the extent of consumption smoothing in the models very well. In particular, the perturbation methods overstate the volatility of consumption while understating the volatility of investment, relative to the volatility of output. This feature is in contrast to the results in the endowment economy model.

More noticeable still is the inability of the perturbation methods to capture the autocorrelation of NX. The BAH and UE setups have low autocorrelation functions, with BAH oscillating between small positive and negative numbers, while UE's autocorrelation function starts at 0.3 and decreases monotonically towards zero. In contrast, the perturbation methods all have a first-order autocorrelation of NX of the order of 0.78, and decreasing to only 0.55 by the fifth-order autocorrelation. Thus, the perturbation methods overstate the persistence of NX. This feature of the solution existed in the endowment economy but has become more noticeable with the introduction of endogenous capital accumulation.

The reason for the poor approximation relates to the earlier discussion of Figure 1. The first-order autocorrelation of the NFA position is 1 (to 2 decimal places – it is in fact strictly below 1 and therefore stationary) in the perturbation methods and only 0.95 and 0.99 for BAH and UE, respectively. The strong nonlinearity in Figure 1 demonstrates how small differences in the autocorrelation of the NFA position can have large effects on the autocorrelation of NX. The crucial parameter capturing the autocorrelation of the NFA position is ψ, the debt elasticity of the interest rate (with $\psi = 0$, the model would display a unit root in the NFA position). In the literature, this parameter has been set arbitrarily to .001 in order to induce the necessary stationarity. The real business cycle moments reveal that the choice of .001 is not quite so innocuous. In fact, in order to better capture the moments of the global model regarding the behavior of NX, a value larger than .001 would have been required. The trade-off though is that the value of ψ also influences precautionary savings and the unconditional mean NFA position. The behavior of these dimensions of the model would be less well captured if ψ was raised.

In addition to the long-run moments under the baseline calibration, Table 4 also presents sensitivity analysis with respect ρ_z and σ_z, the persistence and the standard deviation of the technology innovations, respectively. As in the endowment model, the precautionary savings motive becomes more pronounced as the unconditional volatility of technology increases across the models (except, of course, with the log-linear solution). The perturbation methods, however, again prove to provide a poor approximation at these extreme parameterizations. For example, when $\rho_z = 0.9$, the second-order approximation predicts that the small open economy have builds up a NFA position of 0.33, while the BAH setup implies a negative NFA position of 0.33.

As the variability of technology innovations increase, the relative volatility of the consumption, investment and net exports remain broadly unchanged in the perturbation-based solutions (in the log-linear model, the relative volatilities should be exactly unchanged. Any differences in the results reflect sampling variation). The non-linearities in the second-order approximation and risk-adjusted steady state solution are also not strong enough to show changes in these relative volatilities. This is in contrast to BAH and UE in which the relative volatility of NX is reduced. When the persistence of the technology process increases, we observe the expected pattern in that the autocorrelation of both consumption and investment rise. Where the differences appear are again in the autocorrelation function of NX. The perturbation methods largely display the Fisher separation principle – namely, that the autocorrelation function of net exports is unchanged by the the persistence or variability of the technology shock process. The Fisherian separation is less pronounced in the BAH and UE setups.

We next turn to Figure 4, which presents our analysis of the real business cycle model in the frequency domain. While the BAH and UE specifications generate different spectral plots, the three perturbation-based methods with debt-elastic interest rate function generates spectral plots that are virtually identical. This indicates that the dynamics of the model are not greatly altered, relative to a log-linear approximation around the deterministic steady state, either by taking a second-order approximation of the policy functions or by taking the first-order approximation around the alternative, risk-adjusted steady state.

More importantly, the perturbation methods fail to capture the key features of either the BAH or UE models away from the business cycle frequency. The most noticeable feature of the perturbation methods is the existence of a large weight placed on a very low frequency movement in the NFA position, consumption, NX and capital. While both global models share the feature that much of the action in these variables is at the lower frequencies, the sharp spike in the periodogram at low frequencies in the perturbation methods is not shared by their global counterparts.

The two plots of most interest are NX (left-middle panel) and investment (lower-left panel). For both BAH and UE, NX displays a modestly U-shaped periodogram, implying that net exports are driven by a mixture of very high frequency movements and a low frequency cycle. In sharp contrast, the periodogram for the perturbation methods places almost no weight on high frequency movements and a lot of weight on low frequency movements. This suggests that, in the local approximations, agents do not adjust their NX contemporaneously to a technology innovation, but rather rather adjust the level of NX very gradually to changes the level of technology.

Like NX, investment in the BAH and UE setups display modestly U-shaped periodograms. The perturbation methods however predict something quite different. In particular, the perturbation methods place the majority of weight on cycles at business cycle frequency and above, while displaying very little low frequency movements.

4 Conclusion

In this paper, we have revisited how widely-used stationarity inducing techniques in small open-economy models perform in capturing the dynamics implied by model-based global solution methods. Our findings question the naive use of perturbation methods for the use of studying this class of models. In particular, our results highlight two major challenges for perturbation methods.

First, for these methods to have their best chance to capture the dynamics implied by nonlinear methods, the modelers would need to know how the true, global solution looks. Without having that information in hand, the modeler would be, in a sense, "shooting in the dark." Second, even if the modeler has the true solution in hand, the perturbation methods would, even with the best parametrization, predict significantly different cyclical properties of the model relative to the global solution. While these differences may be less pronounced at business cycle frequencies, the disparities are magnified when the modeler is interested in studying the low frequency behavior of this class of models.

5 Appendix

5.1 Autocorrelation of NFA and NX

$$E\left[(nx_t - \overline{nx})(nx_{t-1} - \overline{nx})\right] = E[nx_t nx_{t-1}] - \overline{nx}^2$$
$$= E[(qb_{t+1} - b_t)(qb_t - b_{t-1})] - \mu_b^2(q-1)^2$$
$$= q^2 E[b_{t+1}b_t] + E[b_t b_{t-1}] - qE[b_t^2] - qE[b_{t+1}b_{t-1}] - \mu_b^2(q-1)^2$$
$$= (\rho\sigma_B^2 + \mu_b^2)(1+q^2) - q(\sigma_B^2 + \mu_b^2) - q\left((1-\rho)\mu_b^2 + \rho(\rho\sigma_B^2 + \mu_b^2)\right) - \mu_b^2(q-1)^2$$
$$= (\rho\sigma_B^2 + \mu_b^2)(1+q^2 - q\rho) - q\left(\sigma_b^2 + (2-\rho)\mu_b^2\right) - \mu_b^2(q-1)^2$$
$$= \sigma_b^2(q^2\rho + \rho - q - q\rho^2)$$

$$\rho(nx) = \frac{q^2\rho + \rho - q - q\rho^2}{1 + q^2 - 2q\rho}. \quad \square \tag{10}$$

5.2 Risk-adjusted steady state

5.2.1 Steady state equations

Risk adjustment effects only the models forward looking equilibrium equations. In the endowment economy model the Euler equation is the only forward looking equation. At the risk-adjusted steady state is

$$0 = 1 - \beta\,(1+r) + M_1. \tag{11}$$

The deterministic counterpart of this equation is $1 = \beta\,(1+r)$. M_1 is therefore the risk adjustment

$$M_1 = -\beta R\sigma^2\,(g_z^c)^2\,\frac{\sigma_z^2}{2} \tag{12}$$

where g_z^c is the slope parameter with respect to the endowment process for the log-linear consumption policy function

$$log\,(c_t) - log\,(c) = g_b^c\,(b_t - b) + g_z^c z_t \tag{13}$$

In the real business cycle model, there are two forward looking equations, the Euler equation for bonds and the Euler equation for capital. The risk-adjusted steady state representation of the bond Euler equation is unchanged from equation (11). The capital Euler equation is

$$0 = 1 - \beta \left(\alpha k^{\alpha-1} l^{1-\alpha} + (1-\delta) \right) + M_2 \tag{14}$$

where

$$M_2 = -\beta \left(\left(\alpha k^{\alpha-1} l^{1-\alpha} \right) \left(\left(g_z^{\Lambda} \right)^2 + 2 g_z^{\Lambda} + 2 g_z^{\Lambda} g_z^{l} + 1 + \left(g_z^{l} \right)^2 \right) + 2\kappa \left(\left(g_z^{k} \right)^2 + g_z^{k} \right) \right) \frac{\eta^2}{2} \tag{15}$$

Again, g_z^i is the slope parameter with respect to the endowment process from variable i's log-linear policy function. $i = \Lambda$ is the stochastic discount factor of the agent.

5.2.2 Solution algorithm

Given the set of equilibrium equations

$$0 = E_t \left(f \left(y_{t+1}, y_t, x_{t+1}, x_t \right) \right), \tag{16}$$

where y is a vector of nonpredetermined and x is a vector of predetermined variables, we can write the corresponding, unknown, policy functions as

$$y_t = g(x_t). \tag{17}$$

In the usual, first-order approximation of a model around its deterministic steady state, the state state vector (y, x) is independent of the dynamics of the model. In contrast, when solving the model around its risk-adjusted steady state, the dynamics of the model in part determine the steady state. This can be seen by observing that the steady state equations in Section 5.2.1, and in particular, the M terms contain first derivatives of the policy function, g_x^y. The steady state and the first-order dynamics of the model must therefore be solved jointly.

Hence, the solution algorithm that we use to approximate equation (17) finds a fixed point between the set of steady state values (y, x) and the set of slope parameters g_x^y.[10] We first solve for the deterministic steady state and corresponding log-linear solution. The values of g_x^y from this solution are used to recalculate the steady state equations in Section 5.2.1. This new steady state implies a new solution to the model first-order dynamics, g_x^y. Iteration continues between g_x^y and

[10] Alternatively, one could use a non-iterative solution algorithm such as de Groot (2013).

(y, x) until convergence is achieved.

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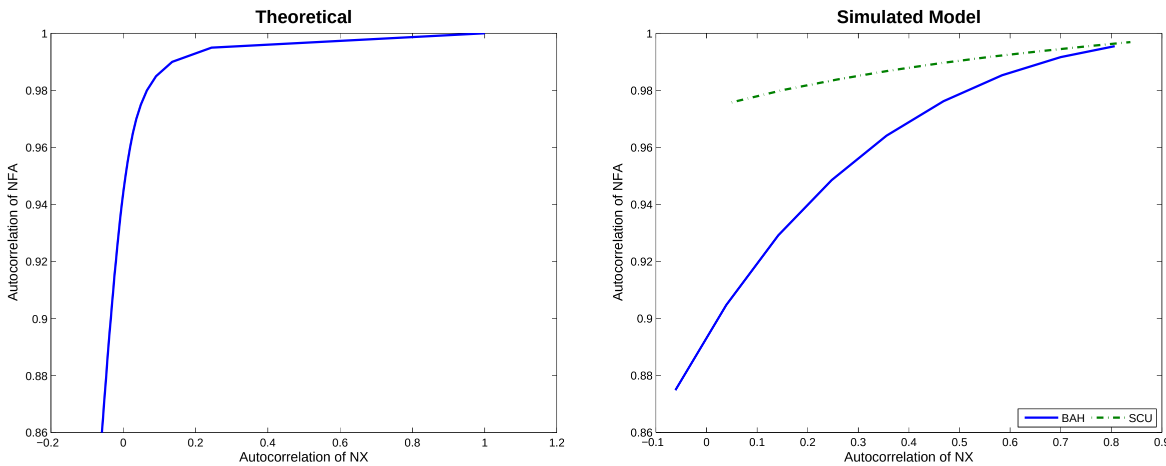
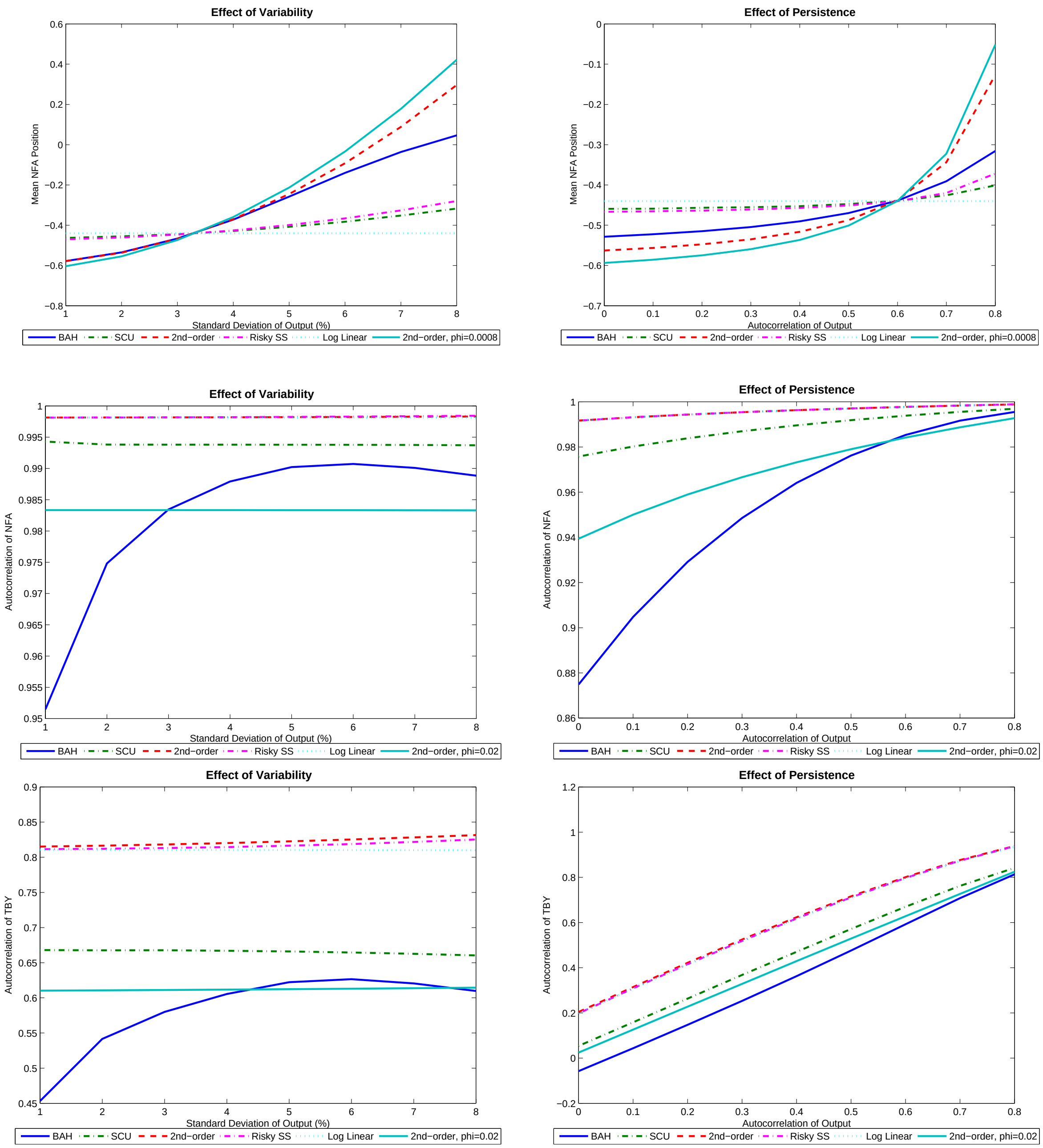
Figure 1: Autocorrelation functions of net exports and net foreign asset positions



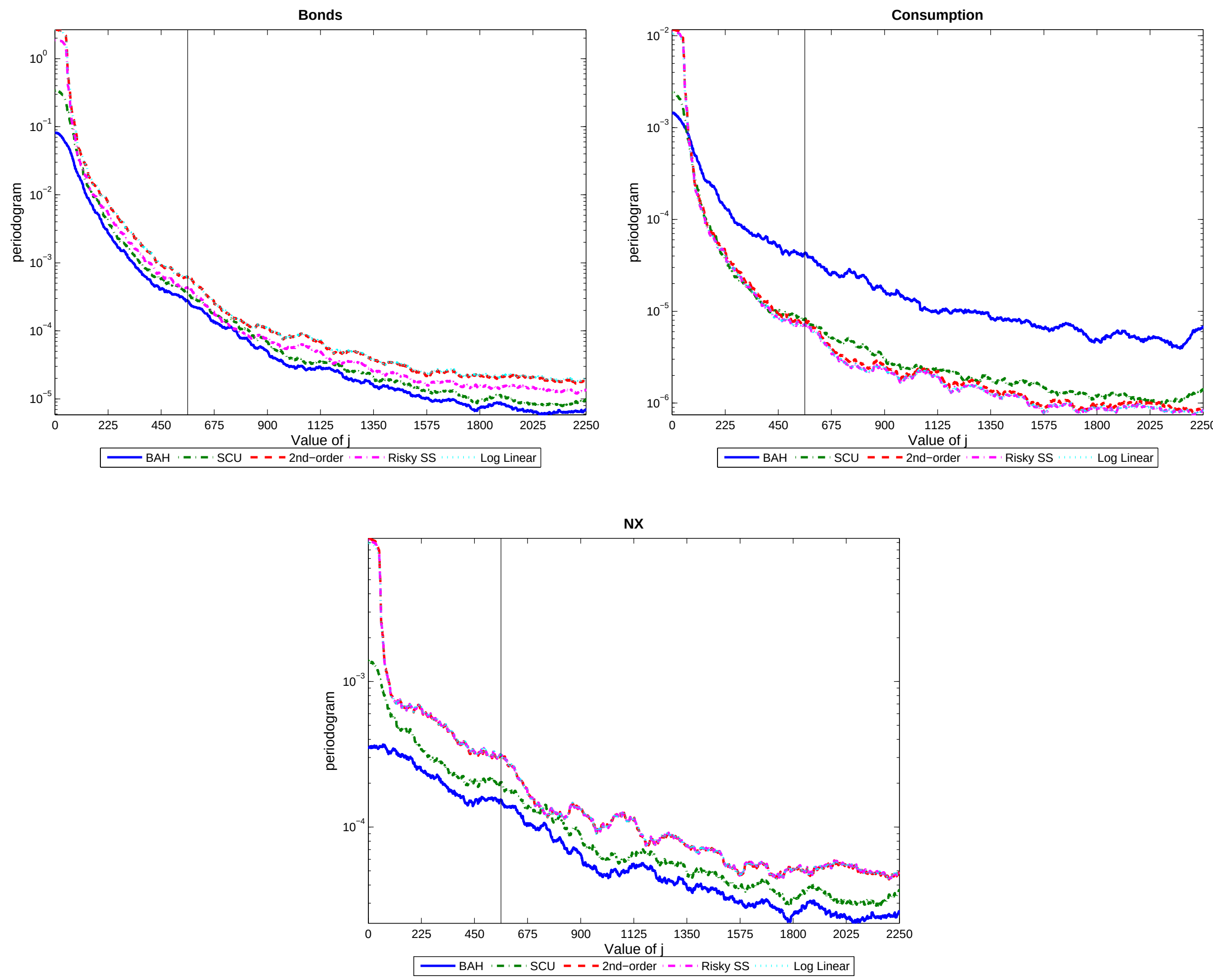
Note: The graph on the left plots the relationship between autocorrelation functions of net exports and net foreign asset positions postulated in Eq. (7). The graph on the right shows the relationship between those variables from the simulated model when the underlying autocorrelation of shock process is changed.

Figure 2: Effects of variability and persistence of output on precautionary demand for foreign assets



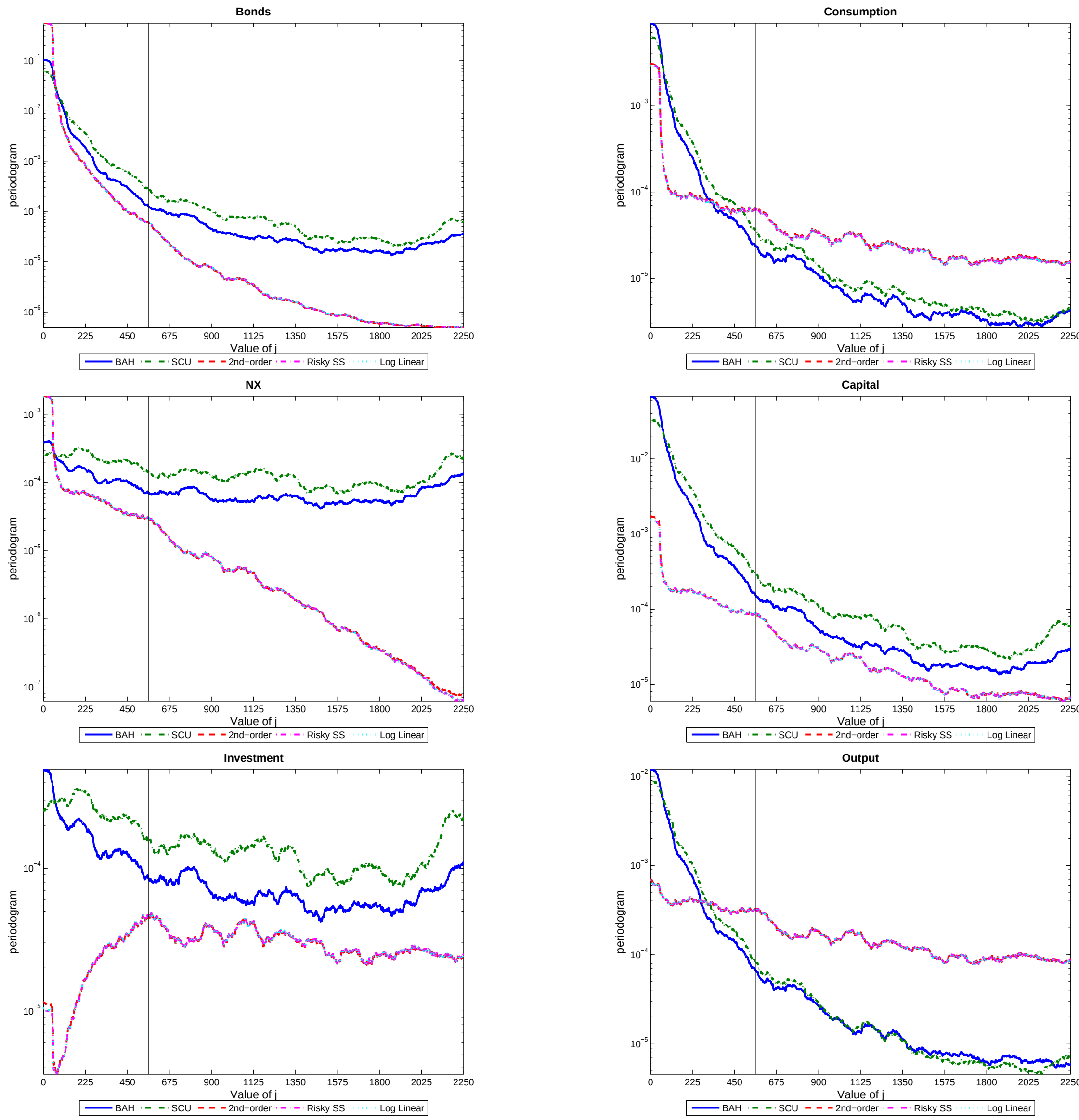
Note: BAH refers to Bewley-Aiyagari-Hugget, UE refers to Uzawa-Epstein.

Figure 3: Macroeconomic aggregates in the frequency domain: one-sector endowment economy



Note: The figure shows the parametric estimates of spectral density for simulated series from four different solution methods: a global approximation with Bewley-Aiyagari-Hugget preferences (BAH), a global approximation with Uzawa-Epstein preferences (UE), a second-order approximation and a log-linearization with debt elastic interest rate functions. The figure shows the estimates as a function of j, where $\omega_j = 2\pi j/T$ and T/j is the period of fluctuations.

Figure 4: Macroeconomic aggregates in the frequency domain: RBC model



Note: These graphs show the parametric estimates of spectral density for simulated macroeconomic aggregates with autocorrelation of the TFP process set to 0.65 and for three different solution methods: a global approximation with Bewley-Aiyagari-Hugget preferences (BAH), a global approximation with Uzawa-Epstein preferences (UE), log-linearization with debt elastic interest rate function. The figure shows the estimates as a function of j, where $\omega_j = 2\pi j/T$ and T/j is the period of fluctuations.

Table 1: Calibration: one-sector endowment economy

Notation	Parameter/Variable	Value
ρ^{BAH}	Rate of time preference in the BAH setup	0.062
ρ^{UE}	Rate of time preference elasticity in the UE setup	0.109
ψ	Elasticity of interest rate with perturbation	0.001
ϕ	Ad-hoc debt limit in the BAH setup	-0.590
R	Gross world interest rate	1.059
y	Mean GDP	1.000
σ_{ε}	Standard deviation of GDP innovations	0.0327
ρ_z	Autocorrelation of GDP	0.597

Note: This table shows the parameter values used in the one-sector endowment economy model. BAH refers to Bewley-Aiyagari-Hugget, and UE refers to Uzawa-Epstein.

Table 2: Long-run moments: one-sector endowment economy

	BAH	SCU	2^{nd}-order	Risky SS	Log-linear
$\mu(b)$	-0.44	-0.43	-0.44	-0.44	-0.44
$\sigma(c)/\sigma(y)$	0.95	0.95	0.93	0.92	0.91
$\sigma(nx)/\sigma(y)$	0.74	0.94	1.20	1.19	1.19
$\sigma(b)/\sigma(y)$	4.24	7.70	14.22	14.29	14.09
$\rho(y,c)$	1.00	1.00	0.98	0.98	0.98
$\rho(y,nx)$	0.31	0.31	0.02	0.02	0.02
$\rho(y,b)$	-0.41	-0.38	-0.42	-0.42	-0.42
$\rho(c)$	0.88	0.97	0.99	0.99	0.99
$\rho(b)$	0.98	0.99	1.00	1.00	1.00
$\rho(nx)$	0.71	0.76	0.78	0.78	0.78
$\rho(nx_t, nx_{t-2})$	0.52	0.62	0.65	0.65	0.64
$\rho(nx_t, nx_{t-3})$	0.41	0.53	0.57	0.57	0.56
$\rho(nx_t, nx_{t-4})$	0.33	0.48	0.52	0.52	0.51
$\rho(nx_t, nx_{t-5})$	0.28	0.44	0.49	0.49	0.48

Note: This table shows the long-run moments of the one-sector endowment economy model. Moments are calculated using HP-filtered series (net exports-output ratio, nx, is based on unfiltered series) from random draws for 50 simulations each solved for 4500 period. BAH refers to Bewley-Aiyagari-Hugget, UE refers to Uzawa-Epstein, and SS refers to the steady-state.

Table 3: Calibration: RBC model

Notation	Parameter/Variable	Value
ρ^{BAH}	Rate of time preference in the BAH setup	0.066
ρ^{UE}	Rate of time preference elasticity in the UE setup	0.157
ψ	Elasticity of interest rate with perturbation	0.001
ϕ	Ad-hoc debt limit in the BAH setup	-0.789
ω	Labor exponent in the utility function	1.65
κ	Investment adjustment cost parameter	0.12
α	Labor share	0.7
δ	Depreciation rate of capital	0.12
R	Gross world interest rate	1.059
σ_{ε}	Standard deviation of GDP innovations	0.013
ρ_z	Autocorrelation of GDP	0.45

Note: This table shows the parameter values used in the RBC model. BAH refers to Bewley-Aiyagari-Hugget, and UE refers to Uzawa-Epstein.

Table 4: Long-run moments: RBC model

	Baseline					$\rho_z = 0.9$					$\sigma_z = 3$ %				
	BAH	SCU	2nd-order	Risky SS	Log-Lin	BAH	SCU	2nd-order	Risky SS	Log-Lin	BAH	SCU	2nd-order	Risky SS	Log-Lin
$\mu(b)$	-0.43	-0.47	-0.42	-0.44	-0.44	-0.33	-0.46	0.33	-0.05	-0.43	-0.31	-0.45	0.01	-0.32	-0.43
$\sigma(y)$	0.03	0.03	0.03	0.03	0.03	0.04	0.04	0.10	0.10	0.10	0.06	0.06	0.07	0.07	0.07
$\sigma(c)/\sigma(y)$	0.78	0.75	0.87	0.86	0.86	0.94	0.90	1.09	1.11	1.07	0.77	0.75	0.87	0.87	0.86
$\sigma(I)/\sigma(y)$	3.56	3.71	3.09	3.13	3.13	2.67	3.33	2.73	2.97	2.95	3.08	3.28	2.99	3.15	3.15
$\sigma(nx)/\sigma(y)$	0.67	0.70	0.76	0.76	0.76	0.59	0.74	0.93	1.00	0.90	0.46	0.55	0.76	0.78	0.75
$\rho(y, nx)$	0.02	0.03	0.02	0.02	0.02	0.02	0.03	-0.02	0.00	0.02	0.02	0.02	0.00	0.02	0.02
$\rho(y, c)$	1.00	1.00	1.01	1.01	1.01	1.01	1.01	1.00	1.00	1.00	1.01	1.01	1.01	1.01	1.01
$\rho(y, I)$	1.02	1.01	1.01	1.01	1.01	1.00	0.98	0.99	0.98	0.99	1.03	1.04	1.01	1.01	1.01
$\rho(y)$	0.65	0.60	0.56	0.56	0.56	0.97	0.96	0.95	0.95	0.95	0.63	0.60	0.56	0.56	0.56
$\rho(c)$	0.78	0.73	0.83	0.83	0.83	0.98	0.96	0.98	0.98	0.98	0.77	0.73	0.83	0.83	0.83
$\rho(I)$	-0.10	0.05	0.10	0.10	0.10	0.33	0.24	0.48	0.46	0.46	0.15	0.09	0.10	0.10	0.10
$\rho(b)$	0.95	0.99	1.00	1.00	1.00	0.99	0.97	1.00	1.00	1.00	0.99	0.99	1.00	1.00	1.00
$\rho(nx)$	-0.32	0.30	0.78	0.78	0.77	0.26	0.20	0.77	0.79	0.75	0.39	0.43	0.79	0.79	0.77
$\rho(nx_t, nx_{t-2})$	0.27	0.16	0.66	0.66	0.66	0.33	0.18	0.66	0.69	0.63	0.23	0.26	0.67	0.68	0.66
$\rho(nx_t, nx_{t-3})$	-0.12	0.12	0.60	0.60	0.60	0.11	0.00	0.60	0.63	0.57	0.13	0.16	0.61	0.62	0.60
$\rho(nx_t, nx_{t-4})$	0.13	0.11	0.57	0.57	0.57	0.20	0.11	0.57	0.60	0.53	0.09	0.12	0.57	0.59	0.57
$\rho(nx_t, nx_{t-5})$	-0.02	0.09	0.55	0.56	0.55	0.06	-0.07	0.55	0.59	0.52	0.08	0.09	0.55	0.57	0.55

Note: This table shows the long-run moments of the RBC model. Moments are calculated using HP-filtered series from random draws for 50 simulations each solved for 4500 period. BAH refers to Bewley-Aiyagari-Hugget, and UE refers to Uzawa-Epstein.