Platform Pricing in Mixed Two-Sided Markets*

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Abstract

In some two-sided markets (2SMs), such as eBay, a user may sometimes buy and sometimes sell. We provide two models for such 2SMs where a participant can appear on both sides, which we call "mixed two-sided markets". In the general model, we show when the platform can price discriminate based on consumers' volume of transactions on each side, the pricing problem is isomorphic to a two-product non-linear pricing problem in a standard "one-sided" market, where the impact of "two-sidedness" only manifests as an upper bound imposed on a consumer's choice set. The basic model focuses on mixed bundling of access to two sides. We show the platform has an incentive to bundle when, without bundling, the demand for each side is on average no more than twice more elastic than the demand for the bundle of two sides. This condition restricts the behavior of four standard price elasticities of demand, and exhibits a "teeterboard" pattern: If the ratio between the price elasticity of demand for the bundle and that for one side is larger, the constraint on the corresponding ratio relevant to the opposite side can be more lax. We also show the incentive to bundle two symmetric sides can be expressed as a definitive threshold on the "degree of mixedness" of the market without bundling. We analyze properties of the optimal platform pricing strategy, which show that when the platform earns a positive economic value (price-cost markup adjusted upwards by network benefits) from one side, the optimal value-to-price ratio on this side will be larger than the inverse of the price elasticity of demand for this side, if and only if the economic value earned on the opposite side does not exceed a threshold. This result starkly contrasts the Lerner formula that applies in non-mixed 2SMs. The higher ratio is necessary to compensate for the additional discounts offered to the consumers who join both sides, whenever the economic value earned from the opposite side alone is insufficient. We also show that the platform's equilibrium profit is strictly increasing in the strength of network effects.

Key Words: mixed two-sided market, platform pricing, price discrimination, bundling

JEL **Classification**: D42, L11, L12, L22.

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1 Introduction

Existing theories of two-sided markets in the literature have mostly focused on classic examples such as credit card, video game and media, where there are good reasons to model the two sides as two *distinct* groups of participants - card holders versus merchants, game players versus developers, and media viewers versus advertisers - because there is little possibility of overlap between these groups (see, for instance, Caillaud and Jullien (2003), Armstrong (2006) and Rochet and Tirole (2003 and 2006)).

However, in many other, perhaps less classic two-sided markets, one participant may act on both sides of the market. Consider for instance the consumer-to-consumer online marketplace mediated by eBay, which clearly exhibits "two-sidedness" as buyers (respectively, sellers) value eBay more when they expect to have access to a larger group of sellers (respectively, buyers) on the platform.[1] More importantly, an eBay user can quite freely buy and sell as he or she pleases, and therefore there exists an overlap between the buyer side and seller side of the market. In fact, the Chinese counterpart of eBay, Taobao.com, has around 1.1% of some 180 million registered users (as of Oct. 2010) who are active in both buying and selling (see Fan, Ju and Xiao(2013)). In telecommunications markets, if we consider the networks as platforms connecting call makers or message senders to receivers, there clearly exist network externalities across the two groups, which indicates two-sidedness. But many (if not most) subscribers both make and receive phone calls, and many both send and receive messages. Therefore the two groups have a conceivably large overlap. Other similar examples include many kinds of financial intermediation where consumers can both buy and sell, or both borrow and lend (such as securities brokerage and social lending), software that allows users to both create and view files in certain formats (such as text-processing software and computer-aided design software), information exchange platforms that allow users to both post (or send) and view messages (such as bulletin boards, online forums, social networking websites and user-generated content platforms), and solar/wind power grids connecting home electric systems that can both draw power from and feed power to the grid.

When there exists a considerably large proportion of participants who act on both sides, one may wonder whether this would change the dynamics between the platform and the two sides, and in particular, whether and how the platform should develop strategies towards such "special" participants. Such problems cannot be properly analyzed using the existing models in the literature, as in these models any such participant would be artificially split into two participants - each on one side of the market only - who make decisions independently of each other. As Armstrong (2006) noted in his discussion of his Proposition 1, "*for the analysis to apply accurately* (to software markets)*, though, there need to be two disjoint groups of agents: those who wish to read files and those who wish to create files. It does not readily apply when most people wish to perform both tasks.*"

In this paper, we provide two models (one basic and the other more general) for two-sided markets where *a consumer can appear on different sides of the market*, which we call **mixed two-sided markets**. If no one can appear on both sides, we call the two-sided market **standard**. Figure 1 illustrates the difference between these two kinds of markets.

[1] This analysis follows the definition of two-sidedness from an "indirect network externality" perspective, as in Armstrong (2006) and Rysman (2009). Alternatively, one could use the definition based on price structures by Rochet and Tirole (2006): If eBay were to charge a fixed total price to the seller and buyer in each transaction, how this price is divided between the two parties will most certainly affect their total volume of transactions.

We show how "mixedness" matters to the platform, present properties of the optimal pricing strategy, and provide conditions on when the platform would find it profitable to use strategies that exploit the mixed nature of the market.

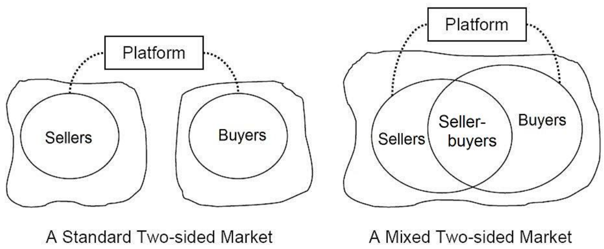

Figure 1: Standard and Mixed Two-Sided Markets

A platform essentially provides two services to consumers – the access to side 1 (e.g. sellers) on the platform, and the access to side 2 (e.g. buyers) on the platform. In a mixed two-sided market, all consumers potentially have the choice to use either one service only, to use both, or to use neither. Therefore the pricing problem that the platform is faced with is *multiproduct* by nature, which means that the platform can employ strategies that are irrelevant in standard two-sided markets. Table 1 shows the choice of pricing strategies by some real-life mixed two-sided platforms.

Table1. Mixed Two-Sided Platforms and Their Pricing Strategies

<table>
<tr><th>Platform</th><th>eBay/Taobao</th><th>Telecom. Network</th><th>Stock Exchange</th><th>Solar/Wind Power Grid</th><th>Acrobat Software</th><th>Zopa.com</th></tr>
<tr><td>Side 1</td><td>Seller</td><td>Caller/Sender</td><td>Seller</td><td>Feeder</td><td>Reader</td><td>Lender</td></tr>
<tr><td>Side 2</td><td>Buyer</td><td>Receiver</td><td>Buyer</td><td>User</td><td>Writer</td><td>Borrower</td></tr>
<tr><td>Strategy Choice</td><td>2-Part Tariff/ Non-linear Pr.</td><td>2-Part Tariff/ Non-linear Pr.</td><td>2-Part Tariff</td><td>2-Part Tariff</td><td>Separate Lump-sum</td><td>Separate 2-Part Tariff</td></tr>
<tr><td>Access to 1</td><td>-</td><td rowspan="2">One common fee</td><td rowspan="2">One common fee</td><td rowspan="2">One common fee</td><td>Fee</td><td>Fee</td></tr>
<tr><td>Access to 2</td><td>-/Fee</td><td>-</td><td>Fee</td></tr>
<tr><td>Usage on 1</td><td>-/Fee</td><td>Fee</td><td>Fee</td><td>Credit</td><td>-</td><td>Fee</td></tr>
<tr><td>Usage on 2</td><td>-</td><td>-/Fee</td><td>Fee</td><td>Fee</td><td>-</td><td>Fee</td></tr>
</table>

As different industries and markets have different business models and/or information availability, not all pricing strategies are feasible in all cases. For instance, Adobe offers its Acrobat Reader software for free, which allows users to read PDF files and hence provides them with an opportunity to interact with people who create these files (i.e. "writers"). On the other hand, the full Acrobat package is sold at a lump-sum price, which can also create/edit PDF files and hence gives writers an opportunity to interact with readers. In either version of the software, there are no additional fees for usage. Zopa.com, a social lending platform facilitating lending and borrowing among individuals, offers different accounts for lenders and borrowers, which involve completely different fees, including membership fees and transaction fees. In telecommunications markets, however, most networks use two-part tariffs where subscribers pay a monthly fixed fee for access to both callers and receivers on the network, plus variable usage fees dependent on the time spent making and/or answering calls, and sometimes discounts are offered on the piece rates of usage to heavy users (e.g. through pre-paid bundles).

We develop two models for different contexts. In the *basic model*, we only consider *mixed bundling* strategies with three prices: one for the access to each side alone, and the third for their bundle (i.e. the access to both sides together). It applies to platforms that are somehow restricted to using lump-sum charges instead of usage-dependent fees, perhaps due to difficulty in monitoring usage (e.g. Acrobat), or when all consumers have the same usage even though two-part tariffs are used[2]. We also extend the basic model to a *general model* that allows the platform to use *multiproduct non-linear pricing*, where the total charge to any consumer is contingent on her individual choice of usage (e.g. number of transactions made) on both sides of the market, which applies when the platform can price based on such usage records (e.g. telecommunications networks).

In the basic model for mixed bundling (section 2), each consumer simply chooses which side(s) to join (if any). We first study the platform's incentive to bundle (i.e. when mixed bundling will strictly dominate *separate pricing* or no bundling[3]), then analyze the optimal way to bundle (i.e. what properties the optimal strategy must have). Our key findings include:

1. The incentive to bundle depends on the sum of two elasticity ratios without bundling, where each ratio is between the *standard price elasticity of demand* for the bundle of two sides and that for one side, with respect to the price for the relevant side. In particular, mixed bundling with a bundle discount (respectively, premium when feasible) is profitable if, without bundling, the demand for each side is *on average no more than twice more elastic* (respectively, on average at least twice more elastic) than the demand for the bundle of two sides.

2. When the two sides are symmetric, the incentive to bundle can be expressed as a definitive threshold on the *degree of mixedness* of the market - the proportion of consumers who join both sides - under optimal separate pricing, where the threshold depends on the ratio between price elasticity of demand for one side (either alone or as part of the bundle) and that for the side alone. A bundle discount (respectively, premium when feasible) is profitable if the degree of mixedness is larger (respectively, smaller) than the threshold.

3. The necessary conditions for the optimal mixed bundling strategy put constraints on the *economic value* that the platform earns from each side. This value is equal to the price-cost markup adjusted upwards by the total external benefits created by each member of this side for the opposite side through network effects. The optimal ratio between economic value and price on each side *no longer* follows the familiar Lerner formula that applies in standard two-sided markets. When the platform earns a positive economic value from one side, the optimal value-to-price ratio may be *larger or smaller than the inverse of the price elasticity of demand* for the relevant

[2]The two-part tariffs studied by Rochet and Tirole (2006) is an example of the latter case, where they assume everyone's volume of transactions is fixed to be either the size of the opposite side, or a given increasing function of that size, instead of chosen according to individual utility maximization. For a monopoly platform in a standard two-sided market, their two-part tariff is equivalent to two lump-sum prices, each for one side. The mixed bundling strategy we study is therefore a multiproduct generalization of their two-part tariffs. More discussion and proof of the equivalence between mixed bundling and two-part tariffs with fixed usage is provided in section 3.5.1.

[3]The separate pricing we use as benchmark for comparison is also the optimal strategy in standard two-sided markets in Armstrong (2006).

side, contingent on how well the platform does on the opposite side. In particular, this ratio will be *larger* than suggested by the Lerner formula, if and only if the economic value earned on the opposite side does *not* exceed a threshold, which in turn depends on the additional discounts offered to consumers joining both sides. The higher value-to-price ratio is necessary exactly to compensate for the additional discounts, whenever the economic value from the opposite side alone is insufficient.

4. The platform's equilibrium profit is strictly increasing in the strength of network effects in both directions across the two sides, at an equal rate that is proportional to the equilibrium size of each side.

In the general model for non-linear pricing (section 3), the platform can price based on a consumer's usage on both sides (if any). In formulating the problem, we show that we can either use the *proportion* of members on each side with whom a consumer chooses to interact to represent usage, or equivalently use the *number* of members chosen on each side. The former method yields an *original representation* of the problem where the parameters of network effects and the endogenously determined market sizes are entangled multiplicatively in a consumer's utility function, as is typical in two-sided market models; whereas the latter method renders a much cleaner *dual representation* where the impact of "two-sidedness" on a consumer's choice only manifests as an upper bound imposed on her choice set, which restricts the number of chosen members of each side to be no larger than the total size of the relevant side. Although the original representation appears quite intractable, the dual representation is isomorphic to a rather standard two-product non-linear pricing problem in a "one-sided" market as in Wilson (1993), except only for the restricted consumer choice set. Therefore we show that:

5. The general model has a solution if and only if the one-sided pricing problem with a *relaxed* choice set in the dual representation has a solution, where the consumers' induced choice of usage does not exceed the size of the relevant side. Whenever this condition holds, the known solution methods in the existing non-linear pricing literature for one-sided markets are directly applicable in solving the general model.

Our models generalize the canonical model of standard two-sided markets by Armstrong (2006), through incorporating features from models of bundling (Long (1984) and Armstrong (2013)) and multiproduct pricing (Wilson (1993)). Consumers are assumed to have heterogeneous types in two dimensions - one for each market side - drawn from a continuous set. Their utility from participating on either side directly increases in the size of the opposite side, which is in turn endogenously determined as the proportion of consumers participating there. The utilities obtained from two sides are additive for each consumer. The platform chooses the best among all feasible pricing strategies in each model to maximize profits. We follow Armstrong (2006) to model demand as functions of utility provided by the platform, unlike in Rochet and Tirole (2006) where demand depends on prices directly. The former approach allows for a much more transparent representation of the demand relationship, crucial for incorporating the multiproduct feature of mixed two-sided markets.

Among the existing works on bundling of two products in "one-sided" markets, Long (1984) and Armstrong (2013) are the strand that uses standard aggregate demand functions and links the incentive to bundle to elasticities of demand. Another strand focuses on the properties of the joint distribution of consumers' valuations for two products (especially their correlation or stochastic dependence), which includes Adams and Yellen (1976), Schmalensee (1984), McAfee, McMillan and Whinston (1989), and Chen and Riordan (2013) (who use copulas instead of distributions, to be more precise), among others. We follow the first strand, and our model is closer to Long (1984) as we focus on a monopoly platform and consumers with additive valuations for two sides. Armstrong (2013) considers much broader situations where there can be separate providers of products as well as substitutability or complementarity in consumption of the bundle. They both find that a bundle discount (respectively, premium) increases profit upon optimal separate pricing (without bundling) if the demand for either product is *less* (respectively, more) elastic than that for the bundle of two products, where the elasticities are defined with respect to "*simultaneous and equal percentage increases in price-cost markups of both products*". This rather unusual definition of elasticity of demand is necessary in Armstrong (2013) particularly because the non-additive valuations he studied do not allow for a simpler representation of the first-order conditions for the optimal separate prices.[4]

As we focus on the case with additivity, we are able to segregate the price effects on different sides in the relevant conditions, and can therefore explain precisely how the incentive to bundle is determined by the *standard* price elasticities of demand for *each* side and those for their bundle. Our result in point 1 presented previously shows a new *"teeterboard" pattern* in the constraint on two sides: What matters is only the *average* behavior of two ratios between elasticities, each relevant to one side; *if the constraint on one of them is tighter, that on the other can be more lax.* For instance, if under optimal separate pricing the demand for side 1 is four-thirds more elastic than the demand for the bundle of two sides (with respect to the price for side 1), then a bundle discount will be profitable as long as the demand for side 2 is no more than four times more elastic than the demand for the bundle (with respect to the price for side 2). It is worth noting that this new pattern does not really rely on two-sidedness, and should be robust in a one-sided context with additive valuations. Our results in points 1 and 2 presented previously are perhaps also easier to test empirically than the existing results, as data for standard elasticities and for the degree of mixedness should potentially be easier to obtain.

Our result in the previous point 3 illustrates the special characteristics of the optimal mixed bundling strategy in mixed two-sided markets. Manelli and Vincent (2006) discuss optimal mixed bundling in one-sided markets with two or more products. Whereas they present results as constraints on the distribution of consumers' valuations that are not directly comparable to our result in terms of elasticities, one sure difference is that, in the first-order conditions, the relevant optimal price-cost markup from each consumer in a one-sided market needs to be adjusted upwards by the positive external benefits that a consumer in a mixed two-sided market creates for the opposite side, and what really matters is the resulting net economic value. Armstrong (2006) actually shows that the optimal (separate) pricing strategy in standard two-sided markets satisfies a modified Lerner formula involving exactly such adjusted markups. In stark contrast, our result in point 3 in turn shows that this modified Lerner formula no longer holds when the two-sided market is mixed.

[4] Long (1984) also needed to define elasticity this way as the assumption of additive valuations (which he called independent component demand) that he claimed to have kept from the Stigler and Adams-Yellen models is not really invoked in his analysis. Otherwise he could also have used the standard price elasticities of demand.

Our general model proposes one way to formulate the general non-linear pricing problem of a mixed two-sided platform within the framework of the familiar one-sided market problem by Wilson (1993), and the result in the previous point 5 provides a necessary and sufficient condition for the equivalence between the solutions to the two problems.

2 A Basic Model for Mixed Bundling

A monopolist platform facilitates interaction between two market sides: $i = 1, 2$. There is a continuum of consumers who may choose to join either one side, both sides, or neither. If a consumer joins side i only, we assume she obtains a total surplus of

$$u_i + t_i$$

where u_i is a *common value* that the platform provides to all side-i members, and t_i represents some *idiosyncratic value* this consumer derives from side i. If a consumer joins both sides, her total surplus is

$$u_1 + t_1 + u_2 + t_2 + u_X \tag{1}$$

where u_X is the extra value (or disutility if negative) provided by the platform in addition to u_1 and u_2. Therefore values obtained from two sides are additive for all consumers. We use $\mathbf{u} \equiv (u_1, u_2, u_X) \in \mathbb{R}^3$ to denote the common values, and $\mathbf{t} \equiv (t_1, t_2) \in \mathbb{R}^2$ to denote the idiosyncratic values, which is also called a consumer's *type*. $\mathbf{u}$ and $\mathbf{t}$ may take positive or negative values. $\mathbf{u}$ is chosen by the platform. $\mathbf{t}$ is exogenously given, and we assume it is distributed on some subset $\mathbb{T}$ of $\mathbb{R}^2$, which satisfies the usual regularity assumption presented in section 3.1.

Not joining either side will yield a total surplus of zero.

2.1 Mixed bundling

If the platform sets $u_X \neq 0$, we say it is using mixed bundling. It means that the platform is intentionally discriminating among the membership choices, and hence a consumer's decision regarding joining one side is linked to that regarding the other side. In this part we focus on a simple mixed bundling strategy as discussed by Armstrong (2013). In section 3.5.1 of the general model, we show this kind of mixed bundling strategy, as well as the two-part tariffs in the real life examples we discuss in section 1, actually can both achieve all feasible pricing strategies in the basic model. Therefore focusing on the former here is without loss of generality, and all the results in section 2 also hold when the platform uses a two-part tariff.

Consider the platform using a *mixed bundling strategy* $\mathbf{p} \equiv (p_1, p_2, p_X) \in \mathbb{R}^3$, where p_i is the fee (or subsidy if negative) charged to a consumer who joins side i, and p_X is the "bundle" discount (or premium if negative) offered to a consumer who joins both sides (who then pays a total of $p_1 + p_2 - p_X$). Enforceability of such a tariff requires the platform's ability to monitor any consumer's membership status, among possibly other things.

The two-sidedness of the market is reflected in the specification of $\mathbf{u}$. Our treatment here mostly follows Armstrong (2006). In particular,

$$\begin{aligned} \text{Side 1 only} &: \quad u_1 \equiv \alpha_1 N_2 - p_1 && \text{(2a)} \\ \text{Side 2 only} &: \quad u_2 \equiv \alpha_2 N_1 - p_2 && \text{(2b)} \\ \text{Both Sides} &: \quad u_b \equiv u_1 + u_2 + u_X = (\alpha_1 N_2 + \alpha_2 N_1) - (p_1 + p_2 - p_X) && \text{(2c)} \end{aligned}$$

which imply $u_X = p_X$.

As in standard two-sided markets, consumers on either side benefit from their interaction with members on the opposite side. For all side-i members, each side-j ($\neq i$) member creates a value of α_i, whose magnitude represents the *strength of network effects*. The total number of consumers who choose to join side i is denoted N_i. Therefore, a consumer who joins only side i obtains a net value of $u_i = \alpha_i N_j - p_i$ from the platform. As the consumers who join both sides get a discount p_X, the extra surplus u_X they get in addition to $u_1 + u_2$ is exactly equal to p_X.

Note however that we allow for negative prices here. Negative prices may emerge in equilibrium because it may be worthwhile for the platform to subsidize participation by some consumers in order to take advantage of the high values created through network effects (as represented by the term $\alpha_i N_j$ in (2)). A negative discount (i.e. a premium) is also possible should it be feasible and the platform find it profitable.

Given $\mathbf{u} = (u_1, u_2, u_X)$ provided by the platform, a consumer chooses the largest amongst the following four options:

$$\{u_1 + t_1, u_2 + t_2, u_1 + t_1 + u_2 + t_2 + u_X, 0\}.$$

We model this discrete choice problem following Armstrong (2013). We define the following demand functions[5]:

$$\begin{aligned} \text{Side 1 only} &: \quad D_1(\mathbf{u}) \equiv \Pr[u_1 + t_1 \geq \max\{u_2 + t_2, u_1 + t_1 + u_2 + t_2 + u_X, 0\}] && \text{(3a)} \\ \text{Side 2 only} &: \quad D_2(\mathbf{u}) \equiv \Pr[u_2 + t_2 \geq \max\{u_1 + t_1, u_1 + t_1 + u_2 + t_2 + u_X, 0\}] && \text{(3b)} \\ \text{Both sides} &: \quad D_b(\mathbf{u}) \equiv \Pr[u_1 + t_1 + u_2 + t_2 + u_X \geq \max\{u_1 + t_1, u_2 + t_2, 0\}] && \text{(3c)} \\ \text{All side-}i &: \quad N_i(\mathbf{u}) \equiv D_i(\mathbf{u}) + D_b(\mathbf{u}) && \text{(3d)} \\ \text{All members} &: \quad N(\mathbf{u}) \equiv D_1(\mathbf{u}) + D_2(\mathbf{u}) + D_b(\mathbf{u}) && \text{(3e)} \end{aligned}$$

The maximized aggregate consumer surplus is then

$$V(\mathbf{u}) \equiv \mathbf{E_t}[\max\{u_1 + t_1, u_2 + t_2, u_1 + t_1 + u_2 + t_2 + u_X, 0\}] \tag{4}$$

[5] Instead of representing demands as functions of prices $\mathbf{p}$, we have chosen to define them as functions of the common values $\mathbf{u}$, which allows for a much more transparent representation of the demand relationship. The cost of transparency, however, is that it also hides some important elements of the model, such as the parameters for network effects, α_1 and α_2, which are not directly shown in the demand functions. For the same reason, features of two-sidedness are not directly shown in some parts of the following analysis.

By the envelope theorem, we have

$$\frac{\partial V}{\partial u_1} = N_1 \tag{5a}$$
$$\frac{\partial V}{\partial u_2} = N_2 \tag{5b}$$
$$\frac{\partial V}{\partial u_X} = D_b \tag{5c}$$

In order to lighten notation, we use superscripts to denote derivatives. In particular, for $i \in \{1, 2, b\}$ and $j \in \{1, 2, X\}$, let[6]

$$\begin{aligned} N_i^j(\mathbf{u}) &\equiv \frac{\partial}{\partial u_j} N_i(\mathbf{u}), \text{ and} \\ D_i^j(\mathbf{u}) &\equiv \frac{\partial}{\partial u_j} D_i(\mathbf{u}). \end{aligned}$$

By the symmetry of 2nd-order derivatives of $V(\mathbf{u})$, we have

$$N_1^2 = N_2^1 \tag{6a}$$
$$N_1^X = D_b^1 \tag{6b}$$
$$N_2^X = D_b^2. \tag{6c}$$

If the platform incurs a fixed cost f_i for each member on side i and there is no other costs, its profit is $N_1(p_1 - f_1) + N_2(p_2 - f_2) - D_b p_X$, which can be written using (2) as a function of $\mathbf{u}$

$$\Pi(\mathbf{u}) \equiv N_1(\alpha_1 N_2 - u_1 - f_1) + N_2(\alpha_2 N_1 - u_2 - f_2) - D_b u_X \tag{7}$$

2.2 Separate pricing

If the platform sets $u_X = 0$, a consumer's decision regarding joining side 1 and that regarding side 2 become independent. To see this, note that when $u_X = 0$, the condition for a consumer to join side i (either alone or in addition to side j), $\max\{u_i + t_i, u_i + t_i + u_j + t_j + u_X\} \geq \max\{u_j + t_j, 0\}$, holds if and only if $u_i + t_i \geq 0$, which has nothing to do with u_j or t_j. This is why we call it a separate pricing strategy.

The demand functions when $u_X = 0$ are defined as follows.

$$\begin{aligned} d_i(u_1, u_2) &\equiv D_i(u_1, u_2, 0) \\ d_b(u_1, u_2) &\equiv D_b(u_1, u_2, 0) \\ n_i(u_i) &\equiv \Pr[u_i + t_i \geq 0] = N_i(u_1, u_2, 0) = d_i(u_1, u_2) + d_b(u_1, u_2). \\ n(u_1, u_2) &\equiv d_1(u_1, u_2) + d_2(u_1, u_2) + d_b(u_1, u_2) \end{aligned}$$

The platform's profit under separate pricing is

$$\pi(u_1, u_2) \equiv n_1(\alpha_1 n_2 - u_1 - f_1) + n_2(\alpha_2 n_1 - u_2 - f_2) \tag{8}$$

[6] In general, the signs of first order derivatives with respect to $\mathbf{u}$ are, for $i, j \in \{1, 2\}$, $i \neq j$, $N_i^i > 0$, $N_i^j > 0$, $N_i^X > 0$, $D_i^i > 0$, $D_i^j < 0$, $D_i^X < 0$, $D_b^i > 0$, $D_b^X > 0$.

The analysis of the optimal separate pricing strategy is the same as in Armstrong (2006).

To the platform, each member of side i brings a direct profit of $p_i - f_i$, and also creates a value of α_j for each member on side j. The net *economic value* that the platform earns from each side-i member, denoted v_i, can be represented as the following

$$v_1 \equiv p_1 - f_1 + \alpha_2 N_2 = (\alpha_1 + \alpha_2)N_2 - u_1 - f_1 \tag{9a}$$

$$v_2 \equiv p_2 - f_2 + \alpha_1 N_1 = (\alpha_1 + \alpha_2)N_1 - u_2 - f_2 \tag{9b}$$

Therefore, we can write the first-order conditions for the optimal separate pricing prices (p_1, p_2) as

$$\frac{p_i - (f_i - \alpha_j n_j)}{p_i} = \frac{v_i}{p_i} = \frac{1}{\epsilon_i^i}, \text{ where } \epsilon_i^i \equiv \frac{n_i'(\alpha_i n_j - p_i) \cdot p_i}{n_i(\alpha_i n_j - p_i)}. \tag{10}$$

Here ϵ_i^i represents the price elasticity of demand for side i under separate pricing, for a given size of the opposite side n_j. This representation is the same as the result in Proposition 1 of Armstrong (2006).

2.3 When to bundle the two sides?

Starting from the optimal separate pricing strategy, if the platform can increase profit by changing u_X, or equivalently by using a bundle discount or premium, we will know for sure that mixed bundling dominates separate pricing.

From (7) we know

$$\frac{\partial}{\partial u_X}\Pi(\mathbf{u}) = N_1^X v_1 + N_2^X v_2 - D_b - D_b^X u_X$$

By (6) we have

$$d_b^i(u_1, u_2) = N_i^X(u_1, u_2, 0) \tag{11}$$

And finally, if (p_1, p_2) is the optimal separate pricing strategy, which induces common values (u_1^S, u_2^S), by (10) we have

$$\begin{aligned} \frac{\partial}{\partial u_X}\Pi(u_1^S, u_2^S, 0) &= d_b^1 \cdot \frac{p_1}{\epsilon_1^1} + d_b^2 \cdot \frac{p_2}{\epsilon_2^2} - d_b \\ &= d_b\left(\frac{d_b^1 p_1/d_b}{\epsilon_1^1} + \frac{d_b^2 p_2/d_b}{\epsilon_2^2} - 1\right) \\ &= d_b\left(\frac{\epsilon_b^1}{\epsilon_1^1} + \frac{\epsilon_b^2}{\epsilon_2^2} - 1\right) \end{aligned}$$

where $\epsilon_i^i \equiv \frac{n_i'(\alpha_i n_j - p_i) \cdot p_i}{n_i(\alpha_i n_j - p_i)}$ as in (10), and $\epsilon_b^i \equiv \frac{d_b^i(\alpha_1 n_2 - p_1, \alpha_2 n_1 - p_2) \cdot p_i}{d_b(\alpha_1 n_2 - p_1, \alpha_2 n_1 - p_2)}$ is the elasticity of demand for the "bundle" of two sides with respect to p_i, provided that the sizes of the two sides are, respectively, n_1 and n_2. As $n_i' > 0$, and $d_b^i > 0$, each ratio $\frac{\epsilon_b^i}{\epsilon_i^i}$ exists and is positive.

Proposition 1 *The platform should use mixed bundling if at the optimal separate pricing strategy*

$$\frac{\epsilon_b^1}{\epsilon_1^1} + \frac{\epsilon_b^2}{\epsilon_2^2} \neq 1. \qquad (12)$$

Moreover, offering a small bundle discount strictly increases profit upon optimal separate pricing if $\frac{\epsilon_b^1}{\epsilon_1^1} + \frac{\epsilon_b^2}{\epsilon_2^2} > 1$*, and charging a small bundle premium (if feasible) strictly increases profit in the opposite case.*

Proposition 1 shows accurately how the incentive of the platform to bundle two sides depends on the four relevant price elasticities of demand in condition (12). The left-hand side of (12) is a joint measure of the magnitude of price elasticities of demand for the bundle of two sides relative to the price elasticities of demand for each side respectively. In words, condition $\frac{\epsilon_b^1}{\epsilon_1^1} + \frac{\epsilon_b^2}{\epsilon_2^2} > 1$ (respectively, $\frac{\epsilon_b^1}{\epsilon_1^1} + \frac{\epsilon_b^2}{\epsilon_2^2} < 1$) roughly means that, the demand for each side is *on average no more than twice more elastic* (respectively, on average at least twice more elastic) than the demand for the bundle of two sides.[7] All elasticities are defined in terms of the price on the relevant side (as indicated by the superscripts in (12)), given the sizes of the two sides at the optimal separate pricing strategy, n_1 and n_2, respectively.

Existing results in the bundling literature have discussed alternative, arguably unusual, definitions of elasticities. Long (1984) (on page S243) and Armstrong (2013) (in his Proposition 1) looked at elasticities of demand with respect to "*simultaneous and equal percentage increases in price-cost markups of both products*". They show that a bundle discount (respectively, premium) is profitable when the so-defined elasticity of demand for the bundle of two products is larger (respectively, smaller) in magnitude than that for one of the two products.

On the contrary, our condition $\frac{\epsilon_b^1}{\epsilon_1^1} + \frac{\epsilon_b^2}{\epsilon_2^2} > 1$, for instance, does not require the demand for either one side to be *less elastic* than that for the bundle, but only asks that the demand for each side is *on average no more than twice more elastic* than that for the bundle, with respect to the respective price for the relevant side. Consider a simple case where $\frac{\epsilon_b^1}{\epsilon_1^1} = \frac{\epsilon_b^2}{\epsilon_2^2} = \frac{2}{3}$, which implies $\left|\epsilon_i^i\right| = 1.5\left|\epsilon_b^i\right|$ for both sides, such that the demand for each side is *more* elastic than that for the bundle (with respect to the price for the relevant side). Using the definition of elasticities by Long (1984) and Armstrong (2013), however, in this situation the elasticity of demand for each side would be 1, and that for the bundle would be $\frac{4}{3}$, which means that, in their terminology, the demand for each side would still be considered *less* elastic than that for the bundle. As different definitions of elasticities are used, these results should not be interpreted as contradicting each other. The conclusion by both their results and our Proposition 1 is actually the same: A positive bundle discount will be profitable.

The results by Long (1984) and Armstrong (2013) are most useful when consumers have non-additive valuations for two products. The bundling part of our model, however, focuses on a special case of theirs, i.e. one with additive valuations (see (1)). In this case, their elasticity of demand for either one product at the optimal separate prices, with respect to "simultaneous and equal percentage increases in price-cost markups of

[7] This interpretation refers to one special case of condition $\frac{\epsilon_b^1}{\epsilon_1^1} + \frac{\epsilon_b^2}{\epsilon_2^2} > 1$: When $\frac{\epsilon_b^i}{\epsilon_i^i} > \frac{1}{2}$ we have $\left|\epsilon_i^i\right| < 2\left|\epsilon_b^i\right|$.

both products", will always be 1; and their condition for a bundle discount (respectively, premium) to be profitable will simply reduce to: *the bundle demand is elastic* (respectively, inelastic), i.e. the so-defined elasticity of demand for the bundle is larger than 1 (respectively, less than 1). Their results tell us no more than this in the situation with additivity.

Our Proposition 1, however, provides further insights and shows precisely how their elasticity concept would compare to 1, which depends on the sum of two standard bundle-to-side demand elasticity ratios, each with respect to the price for the relevant side. Moreover, our condition (12) only restricts the *average* behavior of the two ratios, $\frac{\epsilon_b^1}{\epsilon_1^1}$ and $\frac{\epsilon_b^2}{\epsilon_2^2}$, and therefore illustrates a new "*teeterboard*" pattern between the constraints put on the demand features of two sides. That is, if the constraint on one side is tighter, that on the other can be more lax. For example, if $\frac{\epsilon_b^1}{\epsilon_1^1} = \frac{3}{4}$ such that the demand for the bundle is a little less elastic than the demand for side 1 (with respect to p_1), then the constraint on the demand features of the other side will simply be $\frac{\epsilon_b^2}{\epsilon_2^2} > \frac{1}{4}$, which means a bundle discount will be profitable as long as the demand for side 2 is no more than four times more elastic than the demand for the bundle (with respect to p_2).

As we are using more standard price elasticities of demand, condition (12) is potentially easier to test empirically than the existing results using alternative definitions. Condition (12) also suggests that mixed bundling dominates separate pricing in almost all cases.

Corollary 1 *The platform should use mixed bundling if at the optimal separate pricing strategy*

$$\frac{\epsilon_n^1}{\epsilon_1^1} + \frac{\epsilon_n^2}{\epsilon_2^2} \neq 1, \text{ where } \epsilon_n^i \equiv \frac{\partial n}{\partial u_i}\frac{p_i}{n_i}. \tag{13}$$

Moreover, offering a small bundle discount strictly increases profit upon optimal separate pricing if $\frac{\epsilon_n^1}{\epsilon_1^1} + \frac{\epsilon_n^2}{\epsilon_2^2} < 1$, *and charging a small bundle premium (if feasible) strictly increases profit in the opposite case.*

(All omitted proofs are provided in the Appendix.) An equivalent condition to (12) is (13), although the directions of inequalities need to be reversed. When $\frac{\epsilon_n^1}{\epsilon_1^1} + \frac{\epsilon_n^2}{\epsilon_2^2} < 1$ (respectively, $\frac{\epsilon_n^1}{\epsilon_1^1} + \frac{\epsilon_n^2}{\epsilon_2^2} > 1$), the demand for either side is *on average at least twice more elastic* (respectively, on average no more than twice more elastic) than the demand for the platform's any service(s) (with respect to the price for the relevant side), and offering a small discount (respectively, charging a small premium) is profitable.

2.3.1 Degree of mixedness

We now introduce a measure of how mixed a two-sided market is, **the degree of mixedness**, which is defined as *the proportion of consumers who join both sides amongst all users of the platform.* Formally, given $\mathbf{u}$,

$$M(\mathbf{u}) \equiv \frac{D_b(\mathbf{u})}{N(\mathbf{u})}.$$

$M(\mathbf{u})$ turns out a useful factor in determining the profitability of mixed bundling.

Proposition 2 *Suppose the distribution of types is symmetric with respect to two sides, and the optimal symmetric separate pricing strategy induces common values (u^S, u^S). Then the platform should use mixed bundling if*

$$M(u^S, u^S) \neq 1 - \frac{\epsilon_1^1}{\epsilon_{(1)}^1}, \text{ where } \epsilon_{(1)}^1 \equiv \frac{d_1^1 p_1}{d_1}. \tag{14}$$

Moreover, offering a small bundle discount strictly increases profit if $M(u^S, u^S) > 1 - \frac{\epsilon_1^1}{\epsilon_{(1)}^1}$, and charging a small bundle premium (if feasible) strictly increases profit in the opposite case.

$\epsilon_{(i)}^i$ represents the price elasticity of demand for side i only, given the sizes of the two sides are n_1 and n_2, respectively. When the two market sides are symmetric, Proposition 2 gives a definitive threshold for the degree of mixedness, exceeding which we can be sure that offering a discount is profitable (whereas failing it means a premium is worthwhile). As $\frac{\epsilon_1^1}{\epsilon_{(1)}^1} = \frac{n_1'/n_1}{d_1^1/d_1}$, where all terms are positive, the threshold $(1 - \frac{\epsilon_1^1}{\epsilon_{(1)}^1})$ is strictly smaller than 1.

A rather practical implication of Proposition 2 is that, if the market is symmetric and highly mixed under separate pricing (i.e. $M(u^S, u^S)$ is very close to 1), it is very likely that the platform will find it optimal to use a mixed bundling with a bundle discount. Consider the telecommunications market and stock market, for instance, where it is conceivable that the nature of the market primitives (i.e. the distribution of consumers' type $\mathbf{t}$ in our model) is such that, even under separate pricing, a very high percentage of consumers will still chose to both make and answer calls, or to both buy and sell stocks. $M(u^S, u^S)$ is likely to be very near 1. Therefore a two-part tariff with a common membership fee as shown in Table 1, which implements mixed bundling with a bundle discount, is likely the optimal pricing strategy.

2.4 Optimal mixed bundling

Proposition 3 *The optimal mixed bundling strategy satisfies the following conditions*

$$v_1 = \frac{N_1}{N_1^1} + \frac{D_b^1 p_X - N_2^1 v_2}{N_1^1} \tag{15a}$$

$$v_2 = \frac{N_2}{N_2^2} + \frac{D_b^2 p_X - N_1^2 v_1}{N_2^2} \tag{15b}$$

$$p_X = -\frac{D_b}{D_b^X} + \frac{D_b^1 v_1 + D_b^2 v_2}{D_b^X} \tag{15c}$$

where v_i is the "economic value" that the platform earns from each side-i member as defined in (9).

Proof. These are the first-order conditions of profit maximization under mixed bundling, derived by (7) and (9). Note the economic value that the platform earns from each consumer who joins both sides is $v_b \equiv v_1 + v_2 - p_X$, although only p_X appears in (15c). ■

Consider the impact of the platform raising u_i, which has three marginal effects:

1. Gains on side i: A higher value offered by the platform attracts more consumers. As a result, N_i increases by N_i^i, and from each new member of side i, the platform earns v_i. This is represented on the left-hand side of (15).

2. Losses on side i: As each existing side-i member gets an extra unit of value, raising u_i costs the platform exactly N_i on side i - the first term on the right-hand side of (15).

3. Cross-side effects: As the market is mixed, raising u_i affects the demand for side j (N_j) and that for the bundle of two sides (D_b) as well, which increase by N_j^i and D_b^i, respectively. As each member of side j brings an economic value of v_j, and each consumer of the bundle gets an extra discount p_X, the demand changes in turn result in a *gain* of $N_j^i \cdot v_j$ and a *loss* of $D_b^i \cdot p_X$ to the platform, respectively. Their net impact is represented (as a loss) by the second term on the right-hand side of (15).

These effects exactly cancel one another out at optimality. The analysis of the effects of raising u_X (or equivalently raising p_X) is similar.

To facilitate a clear comparison between the first-order conditions for mixed bundling in (15) and those for separate pricing in (10), we define the following more general price elasticities of demand, given the different market segments of respective sizes N_1, N_2 and D_b. For $i \in \{1, 2\}$ and $j \in \{1, 2, X\}$, let

$$E_i^j \equiv \frac{N_i^j \cdot p_j}{N_i}, \text{ and } E_b^j \equiv \frac{D_b^j \cdot p_j}{D_b}. \tag{16}$$

Corollary 2 *The optimal mixed bundling strategy satisfies the following conditions*

$$\begin{aligned} \frac{p_1 - (f_1 - \alpha_2 N_2)}{p_1} &= \frac{1}{E_1^1} + \frac{D_b^1 p_X - N_2^1 v_2}{p_1 N_1^1} & \text{(17a)} \\ \frac{p_2 - (f_2 - \alpha_1 N_1)}{p_2} &= \frac{1}{E_2^2} + \frac{D_b^2 p_X - N_1^2 v_1}{p_2 N_2^2} & \text{(17b)} \\ 1 &= -\frac{1}{E_b^X} + \frac{D_b^1 v_1 + D_b^2 v_2}{p_X D_b^X}. & \text{(17c)} \end{aligned}$$

Proof. Rewrite (15) using (9) and divide both sides of each equation by the relevant mixed bundling price, and we are done. ■

Proposition 4 *At the optimal mixed bundling strategy, if $v_i > 0$, we have*

$$\left| \frac{p_i - (f_i - \alpha_j N_j)}{p_i} \right| = \left| \frac{v_i}{p_i} \right| > \frac{1}{\left| E_i^i \right|} \textit{ if and only if } v_j < \frac{D_b^i p_X}{N_j^i}. \tag{18}$$

Proof. When $v_i > 0$, as $N_i^i > 0$, $N_j^i > 0$, and $E_i^i = \frac{N_i^i p_i}{N_i}$, we have
$\left|\frac{v_i}{p_i}\right| > \left|\frac{1}{E_i^i}\right| \Leftrightarrow v_i - \frac{Ni}{N_i^i} > 0 \Leftrightarrow D_b^i u_X - N_j^i v_j > 0$ by (15) $\Leftrightarrow v_j < \frac{D_b^i u_X}{N_j^i}$. Done. ■

Condition (18) is in stark contrast to the Lerner formula that applies to the optimal separate pricing strategy as in (10). Under mixed bundling, suppose the platform is earning a positive economic value on side i, then Proposition 4 tells us that the optimal prices $\mathbf{p}$ must yield a "value-to-price ratio" $\left|\frac{v_i}{p_i}\right|$ that is higher than the inverse of the price elasticity of demand for side i, whenever the economic value earned on the opposite side is not sufficient to cover the extra discounts offered to new consumers who join both sides. The latter condition, $v_j < \frac{D_b^i p_X}{N_j^i}$ in (18), means that the cross-side effects of raising u_i discussed in point 3 result in a net loss. The higher ratio on side i is exactly necessary to compensate for this cross-side loss. On the other hand, should the economic value earned on the opposite side be high enough to outweigh the extra discounts paid out, the platform will optimally set a ratio on side i that is lower than what the Lerner formula would suggest.

In summary, whether the optimal value-to-price ratio on one side should be higher or lower than suggested by the Lerner formula depends on the sign of the net cross-side effect, which in turn depends on how well the platform does on the opposite side. As mentioned earlier, the optimal separate pricing strategy in (10) is identical to the optimal strategy of a standard two-sided market in Armstrong (2006). Therefore the contrast between (18) and (10) precisely illustrates the difference between the optimal pricing strategies in mixed and standard two-sided markets. We discuss this contrast again in the numerical example in section 2.5.

2.4.1 Strength of network effects

Proposition 5 *The platform's maximized profit is strictly increasing in both α_1 and α_2, at a rate of $N_1 \cdot N_2$.*

Proof. Rewriting the profit as an explicit function of the parameters α_1 and α_2, and assuming the optimal mixed bundling is $\mathbf{u}^*$, we have

$$\Pi(\mathbf{u}^*, \alpha_1, \alpha_2) = N_1(\alpha_1 N_2 - u_1^* - f_1) + N_2(\alpha_2 N_1 - u_2^* - f_2) - D_b u_X^*.$$

By the envelope theorem

$$\frac{\partial}{\partial \alpha_i}\Pi(\mathbf{u}^*, \alpha_1, \alpha_2) = N_1 \cdot N_2.$$

■

From Proposition 5 we know the platform directly and equally benefits from stronger network effects in either direction across the two sides.

2.5 Numerical example

Suppose the strength of network effects on both sides are $\alpha_1 = \alpha_2 = 0.25$, and the consumer type $\mathbf{t}$ is uniformly distributed on unit square $[-0.3, 0.7] \times [-0.3, 0.7]$. $f_1 = f_2 = 0$. All calculations are done via Scientific WorkPlace 5.0 and the results are presented in Table 2.

Table 2. Numerical Example with $\alpha_i = 0.25$, and $\mathbf{t} \sim \mathbf{U}[-0.3, 0.7] \times [-0.3, 0.7]$

	Separate Pricing		Mixed Bundling	Comment
u_i	-0.233	$>$	-0.463	Negative values to either side
u_X	0		0.735	
u_b	-0.467	$<$	-0.192	
p_i	0.350		0.633	
p_X	0		0.735	M.B. discount large
a_i	0.75	$>$	-0.149	M.B. subsidizes transactions
A	0		0.735	M.B. membership fee high
N_i	0.467	$<$	0.680	
D_i	0.249	$>$	0.007	
D_b	0.218	$<$	0.673	
N	0.716	$>$	0.687	M.B. attracts fewer users overall
$M = \frac{D_b}{N}$	30.4%	$<$	98.0%	M.B. market close to fully mixed
Π	0.327	$<$	0.367	
v_i	0.467	$<$	0.804	
v_b	0.933	$>$	0.872	
$\frac{v_i}{p_i}$	1.333		1.269	
ϵ_i^i or E_i^i	0.75		0.931	
$\frac{1}{\epsilon_i^i}$ or $\frac{1}{E_i^i}$	1.333		1.074	M.B. $\left\lvert\frac{v_i}{p_i}\right\rvert > \left\lvert\frac{1}{E_i^i}\right\rvert$
ϵ_b^i	2.637		$-$	S.P. $\lvert\epsilon_i^i\rvert < 2\lvert\epsilon_b^i\rvert$
$\frac{\epsilon_b^1}{\epsilon_1^1} + \frac{\epsilon_b^2}{\epsilon_2^2}$	7.031		$-$	S.P. $\frac{\epsilon_b^1}{\epsilon_1^1} + \frac{\epsilon_b^2}{\epsilon_2^2} > 1$

Optimal mixed bundling At the optimal mixed bundling strategy, the platform is providing negative common values to all consumers, but the consumers who join both sides get a huge extra value ($u_X = 0.735$). As some consumers have very high idiosyncratic values (t_i may be as high as 0.7), some of them still choose a single side despite the low common value ($u_i = -0.463$). However their proportion is very small. Almost all consumers join both sides, and the degree of mixedness is 98%.

The strategy that achieves this outcome is quite interesting. The platform charges members of each side a price of $p_i = 0.633$, but offers those who join both sides a huge discount of $p_X = 0.735$, which in fact results in a lower net price for the bundle than that for one side ($p_1 + p_2 - p_X = 0.532 < p_i$). This strategy does not seem implementable in practice as each consumer could choose to pay the lower bundle price regardless of which service(s) she will actually use. Of course, this kind of "arbitrage" only works if consumers have free disposal of their "membership status" for each side, or equivalently, have free choice of their volume of transactions on each side, which is not true in the basic model. The bundle of two sides in mixed bundling essentially lumps together N_1 transactions with side 1 and N_2 transactions with side 2, no matter what idiosyncratic values (i.e. type) a consumer has. The general model in section 3, however, relaxes this assumption.

This strategy can also be implemented with an equivalent two-part tariff. The platform can offer a subsidy of $a_i = -0.149$ for each transaction made on either side, and recoup these subsidies through a membership fee $A = 0.735$ that applies to any consumer. The transaction subsidy essentially enhances the positive network effects ($\alpha_i = 0.25$) and makes the market slightly "more two-sided". Enforceability of such a tariff requires the platform's ability to monitor transactions, which in practice is usually not affected by the "freedom of choice" by consumers. This makes it more feasible than the mixed bundling strategy in this particular example.[8]

At the optimal mixed bundling strategy, the platform earns high and relatively "even" economic values from all members, in that $v_i = 0.804$ and $v_b = 0.872$ are rather large and quite close in magnitude, relative to the parameters. The two inequalities in Proposition 4, $\left|\frac{v_i}{p_i}\right| > \frac{1}{|E_i^i|}$ and $v_j < \frac{D_b^i p_X}{N_j^i}$, both hold. Actually, the cross-side effect in point 3 of the discussion of Proposition 3, $D_b^i p_X - N_j^i v_j = 0.123 > 0$. The value-to-price ratio is higher than the inverse of the price elasticity of demand for either side.

Optimal separate pricing The optimal separate pricing strategy involves a relatively low price ($p_i = 0.35$) for either side, which still results in a negative common value ($u_i = -0.233 < 0$) offered to both sides. The platform earns twice more economic value from double-side consumers ($v_b = 0.933$) than from single-side consumers ($v_i = 0.467$). The value-to-price ratio at optimality on side i, $\frac{v_i}{p_i}$, is exactly equal to the inverse of the price elasticity of demand for side i, as the Lerner formula (10) suggests. The equivalent two-part tariff involves a positive transaction fee ($a_i = 0.35$) to both sides, contrary to the optimal transaction subsidy under mixed bundling.

Separate pricing attracts more users overall, but the degree of mixedness is much higher under mixed bundling. At the optimal separate pricing strategy, $\frac{\partial \Pi}{\partial u_X} = 0.218 > 0$, indicating that offering a positive bundle discount (or charging a positive membership fee if a two-part tariff is used) will be profitable. Alternatively, it can be calculated that condition (12) holds with ">" as $\frac{\epsilon_b^1}{\epsilon_1^1} + \frac{\epsilon_b^2}{\epsilon_2^2} = 7.031$; and that condition (14) becomes $30.4\% > 0$ as $\epsilon_{(1)}^1 = \epsilon_1^1 = 0.75$. Indeed we see that the profit of the platform is higher under mixed bundling ($0.367 > 0.327$).

3 A General Model for Non-Linear Pricing

In this section we provide a model where consumers are allowed to choose not only which side(s) to join, but also what proportion of the members on either side with whom they want to interact. We show that when the platform can monitor such choices by consumers and can price based on them, the problem is isomorphic to a two-product non-linear pricing problem in a standard one-sided market.

3.1 Original Representation

Assumption 1 (Type) *The vector of consumers' idiosyncratic values derived from the platform* $\mathbf{t} = (t_1, t_2)$ *follows distribution* G *on support* $\mathbb{T} \subseteq \mathbb{R}^2$, *where* $\mathbb{T}$ is ***weakly convex*** *with* ***full dimension*** *on* $\mathbb{R}^2$.

The total number of members of side i is still denoted N_i. A consumer $\mathbf{t}$ is free to choose what share of all the members on either side (if any) to interact with. The proportion she chooses on side i is denoted $s_i \in [0, 1]$. $s_i = 0$ means that she does not interact with anyone on side i, which in turn means that she does not join side j. $s_i > 0$ then means

[8] More discussion about two-part tariffs and their equivalence to mixed bundling is provided in section 3.5.1.

that the consumer does join side j, and interacts with a positive proportion s_i of members on side i. Therefore the vector $\mathbf{s} \equiv (s_1, s_2) \in [0,1]^2$ shows a consumer's "consumption" choice on the platform. For all side i consumers, the interaction with each member of side $j(\neq i)$ creates a value of α_i.

Therefore, the total value that a consumer $\mathbf{t}$ derives from consumption choice $\mathbf{s}$ on a platform with two sides of sizes (N_1, N_2) is

$$U(\mathbf{s}, \mathbf{t}) \equiv \alpha_1 \cdot N_2 \cdot s_2 + I(s_2) \cdot t_1 + \alpha_2 \cdot N_1 \cdot s_1 + I(s_1) \cdot t_2 \tag{19}$$

where $I(x) = \begin{cases} 1, \text{ if } x>0. \\ 0, \text{ if } x=0. \end{cases}$ Not joining either side will yield a total value $U(\mathbf{0}, \mathbf{t}) = 0$ for all $\mathbf{t}$.

The platform knows the distribution of types in the population, but is unaware of any particular consumer's type. It can however price based on consumption choice through strategy $P(\mathbf{s}) : [0,1]^2 \to \mathbb{R}$.

Given (N_1, N_2) and $P(\cdot)$, consumer $\mathbf{t}$ maximizes utility by choosing optimal consumption

$$\mathbf{s}(\mathbf{t}) = (s_1(\mathbf{t}), s_2(\mathbf{t})) \equiv \arg \max_{\mathbf{s} \in [0,1]^2} U(\mathbf{s}, \mathbf{t}) - P(\mathbf{s}) \tag{20}$$

The size of side i is then "realized" as the number of consumers who choose to interact with a positive share of members on side j

$$N_i = \Pr[s_j(\mathbf{t}) > 0] \tag{21}$$

The platform's profit is then

$$\pi \equiv \mathbf{E}_{\mathbf{t}}[P(\mathbf{s}(\mathbf{t})) - C(\mathbf{s}(\mathbf{t}))] \tag{22}$$

where $C(\mathbf{s}) : [0,1]^2 \to \mathbb{R}^+$ is the cost of providing $\mathbf{s} = (s_1, s_2)$. For instance the cost function may be $C(\mathbf{s}) = c_1 N_1 s_1 + c_2 N_2 s_2$.

The platform chooses the optimal $P(\cdot)$ to maximize profit.

3.2 Dual Representation

There is an alternative way to construct the general model - the dual representation - where we can use the actual *number* of members with whom a consumer interacts to denote a consumption choice, instead of the *proportion* of members.

Denote k_i the number of side-i members that a consumer chooses to interact with. (Therefore, in the notation of the original representation, we have $k_i = N_i \cdot s_i$.) The total value that a consumer $\mathbf{t}$ derives from interaction with $\mathbf{k} = (k_1, k_2) \in \mathbb{R}^{+2}$ members on the platform can be written as

$$\mu(\mathbf{k}, \mathbf{t}) \equiv \alpha_1 \cdot k_2 + I(k_2) \cdot t_1 + \alpha_2 \cdot k_1 + I(k_1) \cdot t_2 \tag{23}$$

And assume the platform can price directly based on $\mathbf{k} = (k_1, k_2)$ by strategy $Q(\mathbf{k}) : \mathbb{R}^{+2} \to \mathbb{R}$.

Given $Q(\cdot)$, consumer $\mathbf{t}$ maximizes utility by choosing optimal consumption

$$\mathbf{k}(\mathbf{t}) = (k_1(\mathbf{t}), k_2(\mathbf{t})) \equiv \arg \max_{\mathbf{k}} \mu(\mathbf{k}, \mathbf{t}) - Q(\mathbf{k}) \tag{24}$$

Therefore the platform chooses the optimal $Q(\cdot)$ to maximize profit[9]

$$\pi = \mathbf{E}_{\mathbf{t}}[Q(\mathbf{k}(\mathbf{t})) - C(\mathbf{k}(\mathbf{t}))]. \tag{25}$$

Inspection of formulae (23) through (25) reveals that they represent something isomorphic to a standard *one-sided* two-product nonlinear pricing problem. As here the utility of any consumer $\mathbf{t}$ does not depend on the total size of either market side, the utility function $\mu(\mathbf{k}, \mathbf{t})$ exhibits no network effects. A consumer here simply chooses the numbers of members of either side to interact with, and the monopolist chooses the optimal pricing strategy based on the numbers chosen by consumers. This is the same as a two-product nonlinear pricing problem formulated in Chapter 13 of Wilson (1993), in the context of a one-sided market.

The final element that the dual representation needs in order to close the two-sided market model is the normalization of the size of side i as the share of consumers who decide to interact with a positive number of members on side j, and to restrict any consumer's consumption choice on either side not to exceed the size of that side. That is,

$$N_i = \Pr[k_j(\mathbf{t}) > 0], j \neq i; i, j \in \{1, 2\}; \text{ and} \tag{26a}$$

$$k_i(\mathbf{t}) \in [0, N_i], \forall \mathbf{t} \in \mathbb{T}. \tag{26b}$$

In the dual representation, the impact of network effects on a consumer's choice is only manifested as a cap on the number of consumers she can interact with on either side, as condition (26b) shows. A larger size of either side gives each consumer a larger set of potential trading partners to choose from. It should therefore be noted that the choice set of the optimal consumption in (24) needs to be restricted to that in condition (26).

3.3 Equivalence

Proposition 6 *The Original Representation ((19) through (22)) of the general model has a solution* ***if and only if*** *the Dual Representation ((23) through (26)) has a solution. If the Dual Representation has an optimal pricing strategy $Q(\cdot)$ which induces two market sides of sizes (N_1, N_2), respectively, then the optimal pricing strategy of the Original Representation is*

$$P(s_1, s_2) = Q(N_1 \cdot s_1, N_2 \cdot s_2) \text{ for any } (s_1, s_2) \in [0, 1]^2.$$

Condition (26) requires that the optimally chosen numbers of members on either side, $k_i(\mathbf{t})$, cannot exceed N_i, the total number of members on that side. If condition (26) holds, we can always convert $k_i(\mathbf{t})$ in the dual representation into $s_i(\mathbf{t})$ in the original representation, and vice versa, and hence the two representations are equivalent.

The two-product nonlinear pricing problem in a one-sided market has been studied in the literature, and there are known methods that can solve it, such as the "parametric-utility" approach by Wilson (1993). The discussion of these methods is beyond the scope of this paper. The following is a summary of the main insights on the optimal pricing strategy and the equilibrium outcome of the market from Wilson (1993), Armstrong (1996) and

[9] Mind the temporary abuse of notations π and C. The technical differences between the cost functions in different representations of the general model are discussed in step 1-2-1 of the proof of Proposition 6 in the Appendix.

Rochet and Chone (1998):

- There is exclusion at the optimal strategy, i.e. some consumers will not be served;
- There is bundling of the two sides, i.e. *ex post* some consumers will choose to participate on both sides and the market will be mixed;
- There is bunching of types, i.e. the platform will not charge all different consumers different total prices. Some consumers of different types will be "forced" to choose the same consumption bundle, and will hence pay the same total price.

3.4 Discussion

An implicit assumption of the general model is that when a consumer chooses the proportions or numbers of members of each side to interact with, *she commits to these choices, and the platform can charge a total fee based on her commitment*, no matter whether these interactions (or the anticipated benefits from them) will be realized or not. Therefore the general model will mostly be applicable to markets where the platform has some kind of "buffer" mechanism for unrealized interactions, such as the "voice mail" system in telecommunications markets.

Not all phone calls made are answered by the people intended - some go to voice mail, some reach the wrong people, and some are simply not answered. An unrealized interaction relevant to the assumption mentioned previously occurs whenever a caller leaves a voice message for a wrong number, which is in turn ignored by the actual receiver. In this situation, the "interaction" intended by the caller only goes half way, and therefore no anticipated benefits will be realized for either the caller or the intended receiver. However, this does not prevent the platform from counting the call towards the caller's usage, and therefore poses no real problem for the platform's pricing mechanism. Notwithstanding the unrealized interactions or benefits, telephone companies are still able to charge based on the number and/or duration of calls made and received, including those that only go "half way".

Many information exchange platforms where users can post and view messages have "buffer" mechanisms similar to the voice mail system that allow messages to be stored for future retrieval. A few such examples are mentioned in section 1, where the general model should also apply. In applications where violation of the assumption mentioned previously is a major concern, however, the general model needs to be modified. In practice, the realization of an interaction requires that both parties involved in the interaction choose each other in their consumption choice. In our model, as each seller (respectively, buyer) creates the same network benefit, say α_i, for all buyers (respectively, sellers) on the opposite side, there is no additional value in "matching" any particular pair of members from two sides. If we were to require that all committed interactions be realized, additional "market clearing" type of conditions, such as $\mathbf{E_t}[N_i \cdot s_i(\mathbf{t})] = \mathbf{E_t}[N_j \cdot s_j(\mathbf{t})]$ or $\mathbf{E_t}[k_i(\mathbf{t})] = \mathbf{E_t}[k_j(\mathbf{t})]$, may be necessary for equilibrium. We have not found a feasible way to solve the general model with such conditions, and this may be a topic of interest for future work.

It is worth mentioning that there is no such problem in our basic model. Like many other models in the literature of two-sided markets, e.g. Armstrong (2006) and Rochet and Tirole (2006), our basic model assumes that joining one side is equivalent to committing to interacting with the whole opposite side, and hence the market always clears.

3.5 Nesting the basic model in the general model

The basic model can be formulated as a special case of the general model. In the original representation of the general model, if we restrict s_i to be either 0 or 1 (instead of any real number in $[0,1]$), we are back to the basic model. Many existing models in the literature on two-sided markets, e.g. Armstrong (2006) and Rochet and Tirole (2003 and 2006), impose this restriction on consumers' choices.

To see this, we use $\lambda_i = 1$ to denote a consumer's choice of joining side i and $\lambda_i = 0$ to denote not joining i.[10] Then the total value that consumer $\mathbf{t}$ derives from "membership choice" $\boldsymbol{\lambda} \equiv (\lambda_1, \lambda_2) \in \{0,1\}^2$ on a platform with two sides of respective sizes (N_1, N_2) is

$$U(\boldsymbol{\lambda}, \mathbf{t}) \equiv \lambda_1(\alpha_1 N_2 + t_1) + \lambda_2(\alpha_2 \cdot N_1 + t_2) \tag{27}$$

Not joining either side will yield a value of $U(\mathbf{0}, \mathbf{t}) = 0$ for all $\mathbf{t}$.

The platform can monitor every consumer's membership choice $\boldsymbol{\lambda}$ and hence can price based on it through strategy $P(\boldsymbol{\lambda}) : \{0,1\}^2 \to \mathbb{R}$. The binary nature of λ_1 and λ_2 means that $P(\cdot)$ can only take on four possible values $P(1,0)$, $P(0,1)$, $P(1,1)$ and $P(0,0)$.

Given (N_1, N_2) and $P(\cdot)$, consumer $\mathbf{t}$ maximizes utility by choosing the optimal $\boldsymbol{\lambda}$

$$\boldsymbol{\lambda}(\mathbf{t}) = (\lambda_1(\mathbf{t}), \lambda_2(\mathbf{t})) \equiv \arg\max_{\boldsymbol{\lambda}} U(\boldsymbol{\lambda}, \mathbf{t}) - P(\boldsymbol{\lambda}) \tag{28}$$

The size of side i is then realized as the number of consumers who choose to join that side

$$N_i = \Pr[\lambda_i(\mathbf{t}) > 0] = \mathbf{E}_{\mathbf{t}}[\lambda_i(\mathbf{t})] \tag{29}$$

The platform then chooses the optimal $P(\cdot)$ to maximize profit

$$\pi \equiv \mathbf{E}_{\mathbf{t}}[P(\boldsymbol{\lambda}(\mathbf{t})) - C(\boldsymbol{\lambda}(\mathbf{t}))] \tag{30}$$

3.5.1 Mixed bundling and two-part tariff as general pricing strategies

It is clear that the mixed bundling strategy we used in the basic model $\mathbf{p} = (p_1, p_2, p_X) \in \mathbb{R}^3$ can be used directly to construct the general pricing strategy $P(\boldsymbol{\lambda}) : \{0,1\}^2 \to \mathbb{R}$ in the previous section in the following way

$$P(\boldsymbol{\lambda}) = \lambda_1 p_1 + \lambda_2 p_2 - \lambda_1 \lambda_2 p_X$$

which implies:

$$P(1,0) = p_1; P(0,1) = p_2; P(1,1) = p_1 + p_2 - p_X; P(0,0) = 0.$$

Now we show that the two-part tariff we discussed in the real-life examples in Table 1 can also construct the general $P(\cdot)$.

Consider the platform using a *two-part tariff* $(A, a_1, a_2) \in \mathbb{R}^3$, where A is a common and fixed "membership" fee (or subsidy if negative) applicable to any user of the platform, and a_i is the marginal "transaction" fee (or subsidy if negative) to a side-i consumer, for every member on the opposite side j (since each i-j member pair interacts once).

[10] $\lambda_i = 1$ (respectively, $\lambda_i = 0$) here corresponds to $s_j = 1$ (respectively, $s_j = 0$) in the general model, where $j \neq i$.

Enforceability of such a tariff requires the platform's ability to monitor any consumer's membership status as well as the size of each side, among possibly other things.

The relationship between mixed bundling strategy (p_1, p_2, p_X) and two-part tariff (A, a_1, a_2) can be summarized as the following

$$p_1 = a_1 N_2 + A; p_2 = a_2 N_1 + A; p_X = A.$$

The two-part tariff (A, a_1, a_2) can be used to construct $P(\cdot)$ in the following way, where N_i is as defined in (29):

$$P(\boldsymbol{\lambda}) = \lambda_1 a_1 N_2 + \lambda_2 a_2 N_1 + (\lambda_1 + \lambda_2 - \lambda_1 \lambda_2) A$$

which implies

$$\begin{aligned} P(1,0) &= a_1 N_2 + A; P(0,1) = a_2 N_1 + A; \\ P(1,1) &= a_1 N_2 + a_2 N_1 + A; P(0,0) = 0. \end{aligned}$$

It is obvious that the two-part tariff (A, a_1, a_2) as well as the mixed bundling strategy $\mathbf{p} = (p_1, p_2, p_X)$ can achieve any pricing strategy $P(\boldsymbol{\lambda})$ with $P(0,0) = 0$, therefore using either of them in the basic model (where we restrict the consumption choice s_i to be either 0 or 1) is without loss of generality.

Long (1984) illustrated the equivalence between mixed bundling and two-part tariffs in a simpler context where N_1 and N_2 in our model are both equal to 1. The rationale is the same here. In practice, there may be situations where one of these two strategies become easier to implement than the other, as we have discussed in the numerical example of section 2.5.[11]

4 Conclusion

The platform pricing problem in mixed two-sided markets has two prominent elements: network effects (from two-sidedness) and multiple products (from mixedness). Our models and results show that: 1) the external benefits that each consumer creates for the opposite side through network effects need to be accounted for in all formulas related to the "net profit" that the platform earns from the consumer; 2) the optimal pricing strategy towards one side does not only depend on how well the platform does on the other side, but also depends on the cross-side effects of demand and profits in the intersection of the two sides; and 3) there are ways that methods and solutions for the one-sided-market context can be applied to the analysis of mixed two-sided markets.

For a mixed two-sided platform in real life, the choice among different theoretical pricing strategies (e.g. that between separate pricing and mixed bundling) may entail much more than just a change in pricing. It may also involve completely different product/system design of the platform. Note that mobile phone networks do not have to offer a SIM card that has both calling and receiving functions. It is technologically feasible to make separate devices that only have either one function. In fact, in the latter half of the last century, telecommunications networks in many countries offered pagers which are mobile devices that can only receive messages. Nor does eBay have to provide all users with both buyer

[11] See Rochet and Tirole (2006) for further discussion of feasibility of different strategies.

and seller services. It is up to the platform whether or not to design the product and/or system to allow the callers to receive calls, to allow the lenders to borrow money, or to allow the buyers to sell. The choice of optimal pricing strategy may be the result of a particular design, and can certainly also be the reason why the platform is designed the way it is.

5 Appendix

Proof of Corollary 1

As $d_b = n_i - d_i$ implies $d_b^i = n_i' - d_i^i$, and $n = d_i + n_j$ implies $\frac{\partial n}{\partial u_i} = d_i^i$, we have

$$\begin{aligned}
\frac{\partial}{\partial u_X}\Pi(u_1^S, u_2^S, 0) &= d_b^1 \cdot \frac{p_1}{\epsilon_1^1} + d_b^2 \cdot \frac{p_2}{\epsilon_2^2} - d_b \\
&= (n_1' - d_1^1)\frac{p_1}{\epsilon_1^1} + (n_2' - d_2^2)\frac{p_2}{\epsilon_2^2} - d_b \\
&= n_1'\frac{p_1}{\epsilon_1^1} - d_1^1\frac{p_1}{\epsilon_1^1} + n_2'\frac{p_2}{\epsilon_2^2} - d_2^2\frac{p_2}{\epsilon_2^2} - d_b \\
&= n_1 + n_2 - d_b - n(\frac{d_1^1 p_1/n}{\epsilon_1^1} - \frac{d_2^2 p_2/n}{\epsilon_2^2}) \\
&= n(1 - \frac{\epsilon_n^1}{\epsilon_1^1} - \frac{\epsilon_n^2}{\epsilon_2^2})
\end{aligned}$$

Therefore

$$\frac{\partial}{\partial u_X}\Pi(u_1^S, u_2^S, 0) \gtreqless 0 \text{ iff } \frac{\epsilon_n^1}{\epsilon_1^1} + \frac{\epsilon_n^2}{\epsilon_2^2} \lesseqgtr 1. \blacksquare$$

Proof of Proposition 2

First consider condition (14) holding with ">". As $M = \frac{d_b}{n} > 1 - \frac{\epsilon_1^1}{\epsilon_{(1)}^1}$ implies $\frac{n_1'/n_1}{d_1^1/d_1} = \frac{\epsilon_1^1}{\epsilon_{(1)}^1} > 1 - \frac{d_b}{n} = \frac{2d_1}{n}$, we have $\frac{n_1'/n_1}{d_1^1} > \frac{2}{n}$. Moreover, $n_1' > 0$, and $d_1^1 > 0$. Therefore

$$\begin{aligned}
\frac{\partial}{\partial u_X}\Pi(u^S, u^S, 0) &= d_b^1 \cdot \frac{n_1}{n_1'} + d_b^2 \cdot \frac{n_2}{n_2'} - d_b \\
&= 2(n_1' - d_1^1)\frac{n_1}{n_1'} - d_b \\
&= n - \frac{2d_1^1}{n_1'/n_1} \\
&> 0.
\end{aligned}$$

And the proof for the opposite case is similar. $\blacksquare$

Proof of Proposition 6

The key elements of the following proof are the relationship between the optimal strategies in different representations, $P(\cdot)$ and $Q(\cdot)$, and that between the optimal consumption choices they induce, $\mathbf{s}(\cdot)$ and $\mathbf{k}(\cdot)$. For completeness, however, many obvious steps implied by piecewise maximization of a consumer's utility are still spelt out.

1. Necessity:

Suppose the Original Representation (henceforth OR) has a solution $P(\cdot)$ which induces optimal consumption $\mathbf{s}(\mathbf{t}) = (s_1(\mathbf{t}), s_2(\mathbf{t}))$ satisfying (20), and incudes $N_i = \Pr[s_j(\mathbf{t}) > 0]$. Note that given any pricing strategy there always exist trivial equilibria where $N_1 = N_2 = 0$; and whenever one of N_i is zero, the other must also be zero in equilibrium. Therefore in this proof we only focus on the more interesting cases where $N_i > 0, i = 1, 2$.

Define $k_i(\mathbf{t}) \equiv N_i \cdot s_i(\mathbf{t}), \forall \mathbf{t}$, and $Q(k_1, k_2) \equiv P(\frac{k_1}{N_1}, \frac{k_2}{N_2}), \forall (k_1, k_2) \in [0, N_1] \times [0, N_2]$.

We need to prove that this $Q(\cdot)$ maximizes the platform's profit in the Dual Representation (henceforth DR), and induces optimal consumption choice $\mathbf{k}(\mathbf{t}) = (k_1(\mathbf{t}), k_2(\mathbf{t}))$ within $[0, N_1] \times [0, N_2]$.

1-1: We first prove that given $P(\cdot)$ and N_i, the $\mathbf{k}(\cdot)$ defined above is optimal in DR.

Suppose it is not, i.e. for some consumer $\mathbf{t}$ there exists $\mathbf{k}'(\mathbf{t}) \in [0, N_1] \times [0, N_2]$, and $\mathbf{k}'(\mathbf{t}) \neq \mathbf{k}(\mathbf{t})$ such that

$$\mu(\mathbf{k}'(\mathbf{t}), \mathbf{t}) - Q(\mathbf{k}'(\mathbf{t})) > \mu(\mathbf{k}(\mathbf{t}), \mathbf{t}) - Q(\mathbf{k}(\mathbf{t})) \tag{31}$$

Suppose further that $\mathbf{k}(\cdot)$ is still optimal for all other consumers, and therefore the unilateral deviation of consumer $\mathbf{t}$ from $\mathbf{k}(\mathbf{t})$ to $\mathbf{k}'(\mathbf{t})$ does not affect the sizes of the two sides (as $\mathbb{T}$ is continuous), and we still have $N_i = \Pr[s_j(\mathbf{t}) > 0] = \Pr[\frac{k_j(\mathbf{t})}{N_j} > 0] = \Pr[k_j(\mathbf{t}) > 0]$.

Let $s_i'(\mathbf{t}) \equiv \frac{k_i'(\mathbf{t})}{N_i}$, $\mathbf{s}'(\mathbf{t}) \equiv (s_1'(\mathbf{t}), s_2'(\mathbf{t}))$, and we know $\mathbf{s}'(\mathbf{t}) \in [0, 1]^2$. From (31) we know:

$$\begin{aligned}
\mu(\mathbf{k}'(\mathbf{t}), \mathbf{t}) - Q(\mathbf{k}'(\mathbf{t})) &= \mu((N_1 s_1'(\mathbf{t}), N_2 s_2'(\mathbf{t})), \mathbf{t}) - Q(N_1 s_1'(\mathbf{t}), N_2 s_2'(\mathbf{t})) \\
&= U(\mathbf{s}'(\mathbf{t}), \mathbf{t}) - P(\mathbf{s}'(\mathbf{t})) \\
&> \mu(\mathbf{k}(\mathbf{t}), \mathbf{t}) - Q(\mathbf{k}(\mathbf{t})) \\
&= U(\mathbf{s}(\mathbf{t}), \mathbf{t}) - P(\mathbf{s}(\mathbf{t}))
\end{aligned}$$

where the second and fourth equalities come from the definitions of $\mu(\cdot)$, $U(\cdot)$, $P(\cdot)$ and $Q(\cdot)$.

This contradicts the assumption that $\mathbf{s}(\mathbf{t})$ is the optimal consumption of consumer $\mathbf{t}$ satisfying (20). Therefore no unilateral deviation by any consumer $\mathbf{t}$ from $\mathbf{k}(\mathbf{t})$ will be profitable, and hence $\mathbf{k}(\cdot)$ is optimal in DR.

1-2-1: We now prove that given $P(\cdot)$, the $Q(\cdot)$ defined above maximizes the platform's profit in DR.

For distinction, define the cost function for DR as $F(\mathbf{k}(\mathbf{t}))$, and still use $C(\mathbf{s}(\mathbf{t}))$ for OR. These cost functions are connected to each other in the following way: *If* $\mathbf{s}(\mathbf{t})$ *induces* $N_i = \Pr[s_j(\mathbf{t}) > 0]$*, and* $k_i(\mathbf{t}) = N_i \cdot s_i(\mathbf{t})$*, then* $F(\mathbf{k}(\mathbf{t})) = F(N_1 s_1(\mathbf{t}), N_2 s_2(\mathbf{t})) = C(\mathbf{s}(\mathbf{t}))$.

Suppose $Q(\cdot)$ does not maximize profit in DR, i.e. there exists some other pricing strategy $Q'(\cdot)$ that is optimal in DR, which induces some consumption choice $\mathbf{k}'(\mathbf{t})$ (not necessarily the same $\mathbf{k}'(\mathbf{t})$ as in step 1-1), and market sizes $N_i' = \Pr[k_j'(\mathbf{t}) > 0]$, such that $N_i' > 0, i = 1, 2$, and $Q'(\cdot)$ satisfies

$$\mathbf{E}_{\mathbf{t}}[Q'(\mathbf{k}'(\mathbf{t})) - F(\mathbf{k}'(\mathbf{t}))] > \mathbf{E}_{\mathbf{t}}[Q(\mathbf{k}(\mathbf{t})) - F(\mathbf{k}(\mathbf{t}))] \tag{32}$$

Let $P'(s_1, s_2) \equiv Q'(N_1' \cdot s_1, N_2' \cdot s_2), \forall (s_1, s_2) \in [0, 1]^2$, and let $s_i'(\mathbf{t}) \equiv \frac{k_i'(\mathbf{t})}{N_i'}$ (not necessarily the same $\mathbf{s}'(\mathbf{t})$ as in step 1-1). Therefore $N_i' = \Pr[k_j'(\mathbf{t}) > 0] = \Pr[s_j'(\mathbf{t}) > 0]$, i.e. $\mathbf{s}'(\mathbf{t})$ induces N_i' in the OR, which implies $F(\mathbf{k}'(\mathbf{t})) = C(\mathbf{s}'(\mathbf{t}))$. Then as $F(\mathbf{k}(\mathbf{t})) = F(N_1 s_1(\mathbf{t}), N_2 s_2(\mathbf{t})) = C(\mathbf{s}(\mathbf{t}))$, where N_i is induced by $\mathbf{s}(\mathbf{t})$, by (32) we have

$$\begin{aligned}
\mathbf{E_t}[Q'(\mathbf{k}'(\mathbf{t})) - F(\mathbf{k}'(\mathbf{t}))] &= \mathbf{E_t}[Q'(N_1's_1'(\mathbf{t}), N_2's_2'(\mathbf{t})) - F(N_1's_1'(\mathbf{t}), N_2's_2'(\mathbf{t}))] \\
&= \mathbf{E_t}[P'(\mathbf{s}'(\mathbf{t})) - C(\mathbf{s}'(\mathbf{t}))] \\
&> \mathbf{E_t}[Q(\mathbf{k}(\mathbf{t})) - F(\mathbf{k}(\mathbf{t}))] \\
&= \mathbf{E_t}[P(\mathbf{s}(\mathbf{t})) - C(\mathbf{s}(\mathbf{t}))]
\end{aligned}$$

If the $\mathbf{s}'(\cdot)$ in the inequality $\mathbf{E_t}[P'(\mathbf{s}'(\mathbf{t})) - C(\mathbf{s}'(\mathbf{t}))] > \mathbf{E_t}[P(\mathbf{s}(\mathbf{t})) - C(\mathbf{s}(\mathbf{t}))]$ is the consumption choice induced by $P'(\cdot)$, we will have found a contradiction to the assumption that $P(\cdot)$ is the optimal pricing strategy in OR, and we will have proved the necessity part of Proposition 6.

1-2-2: Now we prove that the $\mathbf{s}'(\cdot)$ defined in step 1-2-1 is indeed induced by $P'(\cdot)$. This is quite similar to step 1-1.

Suppose it is not, i.e. given $P'(\cdot)$ and $N_i' = \Pr[k_j'(\mathbf{t}) > 0]$, there exists some consumer $\mathbf{t}$ who has a profitable unilateral deviation $\mathbf{s}''(\mathbf{t}) \neq \mathbf{s}'(\mathbf{t})$ that satisfies

$$U(\mathbf{s}''(\mathbf{t}), \mathbf{t}) - P'(\mathbf{s}''(\mathbf{t})) > U(\mathbf{s}'(\mathbf{t}), \mathbf{t}) - P'(\mathbf{s}'(\mathbf{t})) \tag{33}$$

Let $k_i''(\mathbf{t}) \equiv N_i' \cdot s_i''(\mathbf{t})$, and therefore $k_i''(\mathbf{t}) \neq k_i'(\mathbf{t})$. By (33) we have:

$$\begin{aligned}
U(\mathbf{s}''(\mathbf{t}), \mathbf{t}) - P'(\mathbf{s}''(\mathbf{t})) &= U((\frac{k_1''(\mathbf{t})}{N_1'}, \frac{k_2''(\mathbf{t})}{N_2'}), \mathbf{t}) - P'(\frac{k_1''(\mathbf{t})}{N_1'}, \frac{k_2''(\mathbf{t})}{N_2'}) \\
&= \mu(\mathbf{k}''(\mathbf{t}), \mathbf{t}) - Q'(\mathbf{k}''(\mathbf{t})) \\
&> U(\mathbf{s}'(\mathbf{t}), \mathbf{t}) - P'(\mathbf{s}'(\mathbf{t})) \\
&= \mu(\mathbf{k}'(\mathbf{t}), \mathbf{t}) - Q'(\mathbf{k}'(\mathbf{t}))
\end{aligned}$$

where the second and fourth equalities come from the definitions of $\mu(\cdot)$, $U(\cdot)$, $P'(\cdot)$ and $Q'(\cdot)$ in step 1-2-1.

This contradicts the assumption in step 1-2-1 that $\mathbf{k}'(\mathbf{t})$ is the optimal consumption of consumer $\mathbf{t}$ induced by pricing strategy $Q'(\cdot)$ in DR. Therefore no unilateral deviation by any consumer $\mathbf{t}$ from $\mathbf{s}'(\mathbf{t})$ will be profitable, and $\mathbf{s}'(\cdot)$ must be the optimal consumption choice induced by $P'(\cdot)$ in OR.

Therefore in step 1-2-1, we indeed have found a contradiction, and hence we can conclude that given $P(\cdot)$, the $Q(\cdot)$ defined at the beginning of the whole proof does maximize the platform's profit in DR. Combining steps 1-1, 1-2-1 and 1-2-2, we have proved the necessity part of Proposition 6.

2. Sufficiency:

Suppose DR has a solution $Q(\cdot)$ which induces consumption $\mathbf{k}(\mathbf{t}) = (k_1(\mathbf{t}), k_2(\mathbf{t}))$ satisfying (24), and the induced two market sides are of sizes $N_i > 0$ satisfying (26). We need to prove that $P(s_1, s_2) \equiv Q(N_1 \cdot s_1, N_2 \cdot s_2)$, $\forall (s_1, s_2) \in [0, 1]^2$ maximizes profit in OR, and induces optimal consumption $s_i(\mathbf{t}) \equiv \frac{k_i(\mathbf{t})}{N_i}, \forall \mathbf{t}$.

2-1: We first show that given $Q(\cdot)$ and N_i, the $\mathbf{s}(\cdot)$ defined above is the optimal consumption choice in OR.

Suppose it is not, i.e. given $Q(\cdot)$ and $N_i = \Pr[k_j(\mathbf{t}) > 0]$, there exists some consumer

$\mathbf{t}$ who has a profitable unilateral deviation $\mathbf{s}'(\mathbf{t}) \neq \mathbf{s}(\mathbf{t})$ that satisfies

$$U(\mathbf{s}'(\mathbf{t}), \mathbf{t}) - P(\mathbf{s}'(\mathbf{t})) > U(\mathbf{s}(\mathbf{t}), \mathbf{t}) - P(\mathbf{s}(\mathbf{t})) \tag{34}$$

Let $k_i'(\mathbf{t}) \equiv N_i \cdot s_i'(\mathbf{t})$, and therefore $k_i'(\mathbf{t}) \in [0, N_i]$, and $s_i'(\mathbf{t}) = \frac{k_i'(\mathbf{t})}{N_i}$. By (34) we have:

$$\begin{aligned} U(\mathbf{s}'(\mathbf{t}), \mathbf{t}) - P(\mathbf{s}'(\mathbf{t})) &= \mu(\mathbf{k}'(\mathbf{t}), \mathbf{t}) - Q(\mathbf{k}'(\mathbf{t})) \\ &> U(\mathbf{s}(\mathbf{t}), \mathbf{t}) - P(\mathbf{s}(\mathbf{t})) \\ &= \mu(\mathbf{k}(\mathbf{t}), \mathbf{t}) - Q(\mathbf{k}(\mathbf{t})) \end{aligned}$$

where the second and fourth equalities come from the definitions of $\mu(\cdot)$, $U(\cdot)$, $P(\cdot)$ and $Q(\cdot)$.

This contradicts the assumption in that $\mathbf{k}(\mathbf{t})$ is the optimal consumption of consumer $\mathbf{t}$ induced by pricing strategy $Q(\cdot)$ in DR. Therefore no unilateral deviation by any consumer $\mathbf{t}$ from $\mathbf{s}(\mathbf{t})$ will be profitable, and $\mathbf{s}(\cdot)$ must be the optimal consumption choice in OR.

2-2-1: We now show that given $Q(\cdot)$, the $P(\cdot)$ defined above maximizes the platform's profit in OR.

Suppose it does not, i.e. there exists some other pricing strategy $P'(\cdot)$ that is optimal in OR, which induces some consumption choice $\mathbf{s}'(\mathbf{t})$ (not necessarily the same $\mathbf{s}'(\mathbf{t})$ as in step 2-1), and market sizes $N_i' = \Pr[s_j'(\mathbf{t}) > 0]$, such that $N_i' > 0, i = 1, 2$, and $P'(\cdot)$ satisfies

$$\mathbf{E}_{\mathbf{t}}[P'(\mathbf{s}'(\mathbf{t})) - C(\mathbf{s}'(\mathbf{t}))] > \mathbf{E}_{\mathbf{t}}[P(\mathbf{s}(\mathbf{t})) - C(\mathbf{s}(\mathbf{t}))] \tag{35}$$

Let $k_i'(\mathbf{t}) \equiv N_i' \cdot s_i'(\mathbf{t}), \forall \mathbf{t}$ (not necessarily the same $\mathbf{k}'(\mathbf{t})$ as in step 2-1), and $Q'(k_1, k_2) \equiv P'(\frac{k_1}{N_1'}, \frac{k_2}{N_2'})$, $\forall (k_1, k_2) \in [0, N_1'] \times [0, N_2']$. By definition, $F(\mathbf{k}'(\mathbf{t})) = C(\mathbf{s}'(\mathbf{t}))$. According to the definition of $\mathbf{s}(\mathbf{t})$ at the beginning of the sufficiency part, $N_i = \Pr[k_j(\mathbf{t}) > 0] = \Pr[s_j(\mathbf{t}) > 0]$, therefore $F(\mathbf{k}(\mathbf{t})) = F(N_1 s_1(\mathbf{t}), N_2 s_2(\mathbf{t})) = C(\mathbf{s}(\mathbf{t}))$. Then by (35) we have

$$\begin{aligned} \mathbf{E}_{\mathbf{t}}[P'(\mathbf{s}'(\mathbf{t})) - C(\mathbf{s}'(\mathbf{t}))] &= \mathbf{E}_{\mathbf{t}}[Q'(\mathbf{k}'(\mathbf{t})) - F(\mathbf{k}'(\mathbf{t}))] \\ &> \mathbf{E}_{\mathbf{t}}[P(\mathbf{s}(\mathbf{t})) - C(\mathbf{s}(\mathbf{t}))] \\ &= \mathbf{E}_{\mathbf{t}}[Q(\mathbf{k}(\mathbf{t})) - F(\mathbf{k}(\mathbf{t}))] \end{aligned}$$

If the $\mathbf{k}'(\cdot)$ in the inequality $\mathbf{E}_{\mathbf{t}}[Q'(\mathbf{k}'(\mathbf{t})) - F(\mathbf{k}'(\mathbf{t}))] > \mathbf{E}_{\mathbf{t}}[Q(\mathbf{k}(\mathbf{t})) - F(\mathbf{k}(\mathbf{t}))]$ is the consumption choice induced by $Q'(\cdot)$, we will have found a contradiction to the assumption that $Q(\cdot)$ is the optimal pricing strategy in DR, and we will have proved the sufficiency part of Proposition 6.

2-2-2: Now we prove that the $\mathbf{k}'(\cdot)$ defined in step 2-2-1 is indeed induced by $Q'(\cdot)$. This is quite similar to step 2-1.

Suppose it is not, i.e. given $Q'(\cdot)$ and $N_i' = \Pr[s_j'(\mathbf{t}) > 0]$, there exists some consumer $\mathbf{t}$ who has a profitable unilateral deviation $\mathbf{k}''(\mathbf{t}) \neq \mathbf{k}'(\mathbf{t})$ that satisfies

$$\mu(\mathbf{k}''(\mathbf{t}), \mathbf{t}) - Q'(\mathbf{k}''(\mathbf{t})) > \mu(\mathbf{k}'(\mathbf{t}), \mathbf{t}) - Q'(\mathbf{k}'(\mathbf{t})) \tag{36}$$

Let $s_i''(\mathbf{t}) \equiv \frac{k_i''(\mathbf{t})}{N_i'}$, and therefore $s_i''(\mathbf{t}) \neq s_i'(\mathbf{t})$. By (36) we have:

$$\begin{aligned}
\mu(\mathbf{k}''(\mathbf{t}),\mathbf{t}) - Q'(\mathbf{k}''(\mathbf{t})) &= \mu((N_1's_1''(\mathbf{t}), N_2'k_2''(\mathbf{t})),\mathbf{t}) - Q'(N_1's_1''(\mathbf{t}), N_2'k_2''(\mathbf{t})) \\
&= U(\mathbf{s}''(\mathbf{t}),\mathbf{t}) - P'(\mathbf{s}''(\mathbf{t})) \\
&> \mu(\mathbf{k}'(\mathbf{t}),\mathbf{t}) - Q'(\mathbf{k}'(\mathbf{t})) \\
&= U(\mathbf{s}'(\mathbf{t}),\mathbf{t}) - P'(\mathbf{s}'(\mathbf{t}))
\end{aligned}$$

where the second and fourth equalities come from the definitions of $\mu(\cdot)$, $U(\cdot)$, $P'(\cdot)$ and $Q'(\cdot)$ in step 2-2-1.

This contradicts the assumption in step 2-2-1 that $\mathbf{s}'(\mathbf{t})$ is the optimal consumption of consumer $\mathbf{t}$ induced by pricing strategy $P'(\cdot)$ in OR. Therefore no unilateral deviation by any consumer $\mathbf{t}$ from $\mathbf{k}'(\mathbf{t})$ will be profitable, and $\mathbf{k}'(\cdot)$ must be the optimal consumption choice induced by $Q'(\cdot)$ in DR.

Therefore in step 2-2-1, we indeed have found a contradiction, and hence we can conclude that given $Q(\cdot)$, the $P(\cdot)$ defined at the beginning of the sufficiency part does maximize the platform's profit in OR. Combining steps 2-1, 2-2-1 and 2-2-2, we have proved the sufficiency part of Proposition 6.■

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