Myths and Facts about Panel Unit Root Tests*

Joakim Westerlund† — Lund University, Sweden
Jörg Breitung — University of Bonn, Germany

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Abstract

This paper points to some of the common myths and facts that have emerged from 20 years of research into the analysis of unit roots in panel data. Some of these are well-known, others are not. But they all have in common that if ignored the effects can be very serious. This is demonstrated using both simulations and theoretical reasoning.

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1 Introduction

Starting with the working paper versions of Quah (1994) and Breitung and Meyer (1994) that were available already in 1989, the literature concerned with the analysis of unit roots in panel data covers more than 20 years. While during the first decade the topic was rather peripheral, it has by now become a very active research area, see for example Choi (2006) and Breitung and Pesaran (2008) for recent surveys of the literature. Today panel unit root tests are standard econometric tools within most fields of empirical economics, especially in macroeconomics and financial economics, and some are now available in commercial software packages such as EViews and STATA.

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†Corresponding author: Department of Economics, Lund University, P. O. Box 7082, S-220 07 Lund, Sweden. Telephone: +46 46 222 8670, Fax: +46 46 222 4118, E-mail address: joakim.westerlund@nek.lu.se.

In the beginning when the panel unit root literature was still in its infancy econometricians tended to view extensions of the conventional unit root analysis to panel data as a rather straightforward and less exciting exercise. However, it has since then become clear that this is not the case. Indeed, subsequent work has revealed a number of surprising results and it seems fair to say that adapting conventional unit root analysis to a panel data framework has revealed fundamental differences in the way statistical inference with non-stationary data is performed.

In line with this development the current paper argues that extensions of existing time series unit root tests to panels can sometimes be deceptive in their simplicity. In particular, we argue that the usual practice of looking at the testing problem from a time series perspective gives rise to a number of myths, and increases the risk of overlooking important facts, some of which are well-known, others are not. However, they all share the feature that if ignored the effects upon analysis can be dramatic, with deceptive inference as a result. In fact, as we shall see, in most cases ignorance will actually cause the panel unit root statistic to become divergent, thus leading to a complete breakdown of the whole test procedure. Proper understanding of these myths and facts is therefore key in any research with non-stationary panel data.

The plan of the paper is the following. Section 2 focuses on the simplest case without any deterministic terms, short-run dynamics or cross-sectional dependence. Although admittedly very restrictive, this setup allows us to focus on some of the most basic differences between the analysis of time series and panel data. In Section 3 we generalize the setup of Section 2 to allow for deterministic constant and trend terms. The analysis reveal that this small change has major implications for the asymptotic analysis. Models with short-run dynamics are considered in Section 4 and in Section 5 we address the problems that arise when the cross-sectional units are no longer independent. Section 6 offers some concluding remarks.

2 The simplest case

Consider the double indexed variable y_{it}, observable for $t = 1, ..., T$ time periods and $i = 1, ..., N$ cross-sectional units. Initially we will assume that y_{it} has no deterministic part, so that

$$y_{it} = y_{it}^s, \tag{1}$$

where y_{it}^s is the stochastic part of y_{it}, which is assumed to evolve according to the following first-order autoregressive (AR) process:

$$y_{it}^s = \rho_i y_{it-1}^s + \varepsilon_{it}, \quad (2)$$

or, equivalently,

$$\Delta y_{it} = (\rho_i - 1) y_{it-1} + \varepsilon_{it} = \alpha_i y_{it-1} + \varepsilon_{it}. \quad (3)$$

In this section we assume that the error ε_{it} is mean zero and independent across both i and t. To make life even simpler, we assume that the errors are homoscedastic so that $E(\varepsilon_{it}^2) = \sigma^2$ for all i and t. Note that while unduely restrictive for most practical purposes, this data generating process has the advantage of being simple and illustrative.

The null hypothesis of interest is

$$H_0 : \alpha_i = 0 \text{ for all } i,$$

which corresponds to a fully non-stationary panel. As for the alternative hypothesis, we will consider two candidates, H_{1a} and H_{1b}. The first is specified as

$$H_{1a} : \alpha_i = \alpha < 0 \text{ for all } i,$$

and corresponds to a fully stationary panel with the same degree of mean reversion for all units. It is therefore quite restrictive. The second alternative is more relaxed. It reads

$$H_{1b} : \alpha_i < 0 \text{ for } i = 1, ..., N_1 \text{ with } \frac{N_1}{N} \to \delta_1 > 0 \text{ as } N_1, N \to \infty,$$

which corresponds to a mixed panel with δ_1 being the limiting fraction of stationary units. Note that in this formulation, there are no homogeneity restrictions with regards to the degree of mean reversion. Note also that at this point we make no assumptions concerning the remaining $N - N_1$ slopes, $\alpha_{N_1+1}, ..., \alpha_N$, which may all be zero, negative or a mixture of both. However, we do require that $\delta_1 > 0$, as otherwise the panel would escape stationarity as N_1, $N \to \infty$.

The two alternative hypotheses H_{1a} and H_{1b} are chosen to match the two tests that will be of primary interest in this paper, the Levin *et al.* (2002) test and the Im *et al.* (2003) test, henceforth LLC and IPS, respectively.

Before considering these tests, however, it is useful to introduce some notation. In particular, we define $M = \sum_{i=1}^N M_i$, where

$$M_i = \begin{pmatrix} M_{11i} & M_{12i} \\ \cdot & M_{22i} \end{pmatrix} = \sum_{t=2}^T \begin{pmatrix} (\Delta y_{it})^2 & y_{it-1}\Delta y_{it} \\ \cdot & y_{it-1}^2 \end{pmatrix}$$

is the non-normalized moment matrix of the variables contained in the regression in (3), whose asymptotic counterpart is given by

$$M_i^\circ = \begin{pmatrix} M_{11i}^\circ & M_{12i}^\circ \\ \cdot & M_{22i}^\circ \end{pmatrix} = \int_0^1 \begin{pmatrix} \sigma^2 & W_i(s)dW_i(s) \\ \cdot & W_i(s)^2 ds \end{pmatrix},$$

where $W_i(s)$ is a standard Brownian motion on $s \in [0, 1]$. In particular, it holds that

$$\begin{pmatrix} \frac{1}{T}M_{11i} & \frac{1}{T}M_{12i} \\ \cdot & \frac{1}{T^2}M_{22i} \end{pmatrix} \Rightarrow \sigma^2 M_i^\circ$$

as $T \to \infty$, where the symbol $\Rightarrow$ signifies weak convergence.

The results reported in this paper are derived using either the joint limit method wherein $N, T \to \infty$ simultaneously, or the sequential limit method wherein one of the indices is passed to infinity before the other, see Phillips and Moon (1999). In any case, since the purpose here is more to illustrate rather than to prove, details that are not essential for the understanding of the main point will be omitted. The derivations will therefore not be complete, and readers are referred to the relevant original works for a more detailed treatment.

Having introduced the main notation, we now go on to discuss the IPS and LLC tests. With no serial correlation or heteroskedasticity, and no deterministic constant or trend terms, the Levin and Lin (1992) statistic is given by

$$\tau_{LLC} = \frac{M_{12}}{\hat{\sigma}\sqrt{M_{22}}} = \hat{\alpha}\,\frac{\sqrt{M_{22}}}{\hat{\sigma}},$$

where $\hat{\sigma}^2 = \frac{1}{NT}(M_{11} - \hat{\alpha}\,M_{12})$ with $\hat{\alpha} = M_{12}/M_{22}$ being the least squares estimator of α, whose standard error is given by $\hat{\sigma}/\sqrt{M_{22}}$. Note that although in this setting the Levin and Lin (1992) statistic is the same as the LLC statistic that assumes no deterministic component and no short-run dynamics, at times it will be important to keep the distinction, as this similarity is not always going to hold when we go on to discuss more general models.

The IPS test is given by

$$\tau_{IPS} = \frac{\sqrt{N}(\overline{\tau} - E(\tau))}{\sqrt{\mathrm{var}(\tau)}},$$

where $\overline{\tau} = \frac{1}{N}\sum_{i=1}^{N}\tau_i$ and τ_i is the usual Dickey and Fuller (1979), or DF, test statistic,

$$\tau_i \;=\; \frac{M_{12i}}{\hat{\sigma}_i\sqrt{M_{22i}}} \;=\; \hat{\alpha}_i\,\frac{\sqrt{M_{22i}}}{\hat{\sigma}_i}$$

with an obvious definition of $\hat{\sigma}_i^2$ and $\hat{\alpha}_i$. It is well-known that

$$\tau_i \;\Rightarrow\; \frac{M^\circ_{12i}}{\sqrt{M^\circ_{22i}}}$$

as $T \to \infty$. The constants $E(\tau)$ and $\mathrm{var}(\tau)$ are simply the mean and variance of this limiting distribution. Note that since M_i° is identically distributed, $E(\tau)$ and $\mathrm{var}(\tau)$ do not need to carry an i index.

Fact 1: The IPS and LLC statistics are standard normally distributed as $N \to \infty$.

In order to establish the asymptotic normality of τ_{LLC} and τ_{IPS} we invoke two of the most important tools of the analysis of non-stationary panel data, the weak law of large numbers and the Lindeberg–Levy central limit theorem.

Consider first the LLC statistic, which can be written as

$$\tau_{LLC} \;=\; \frac{M_{12}}{\hat{\sigma}\sqrt{M_{22}}} \;=\; \frac{\frac{1}{T\sqrt{N}}M_{12}}{\hat{\sigma}\sqrt{\frac{1}{NT^2}M_{22}}}.$$

We begin by analyzing the denominator under H_0, which by the law of large numbers as $N \to \infty$ becomes

$$\begin{aligned}\frac{1}{NT^2}M_{22} \;&\xrightarrow{p}\; \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\frac{1}{T^2}\,E(M_{22i}) \;=\; \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\frac{1}{T^2}\sum_{t=2}^{T}E(y_{it-1}^2)\\ &=\; \sigma^2\frac{1}{T^2}\sum_{t=2}^{T}t \;=\; \sigma^2\,\frac{T+1}{2T},\end{aligned}$$

where $\xrightarrow{p}$ signifies convergence in probability. Similarly, $\frac{1}{NT}M_{12} \xrightarrow{p} 0$ and $\frac{1}{NT}M_{11} \xrightarrow{p} \sigma^2$ as $N \to \infty$, from which we deduce that $\hat{\alpha} \xrightarrow{p} 0$ and $\hat{\sigma}^2 \xrightarrow{p} \sigma^2$.

Moreover,

$$\begin{aligned}\mathrm{var}\left(\frac{1}{T}M_{12i}\right) \;&=\; \frac{1}{T^2}\sum_{t=2}^{T}\mathrm{var}(y_{it-1}\Delta y_{it}) \;=\; \sigma^2\frac{1}{T^2}\sum_{t=2}^{T}\mathrm{var}(y_{it-1}) \;=\; \sigma^2\frac{1}{T^2}\sum_{t=2}^{T}E(M_{22i})\\ &=\; \sigma^4\,\frac{T+1}{2T}.\end{aligned}$$

In view of this result and the assumed independence across i, we have that by the Lindeberg–Levy central limit theorem as $N \to \infty$

$$\frac{1}{T\sqrt{N}} M_{12} \;=\; \frac{1}{T\sqrt{N}} \sum_{i=1}^{N} M_{12i} \;\stackrel{d}{\to}\; \sigma^2 \sqrt{\frac{T+1}{2T}} \mathcal{N}(0,1),$$

where $\stackrel{d}{\to}$ denotes convergence in distribution. Thus, by putting everything together we get

$$\tau_{LLC} \;=\; \frac{\frac{1}{T\sqrt{N}} M_{12}}{\hat{\sigma}\sqrt{\frac{1}{T^2 N} M_{22}}} \;\stackrel{d}{\to}\; \mathcal{N}(0,1).$$

Note that this result holds for any T. Hence, the asymptotic normality of the LLC statistic does not require $T \to \infty$. However, if individual specific parameters relating to for example deterministic terms or short-run dynamics are introduced, then this is no longer true. The reason is that consistent estimation of these parameters requires $T \to \infty$, see for example Harris and Tzavalis (1999) and LLC.

In a similar manner it can be shown that τ_{IPS} also has a standard normal limiting distribution as $N \to \infty$ with T held fixed. In particular, as pointed out by IPS as long as $E(\tau)$ and $\text{var}(\tau)$ are evaluated for a finite T, then by the Lindeberg–Levy central limit theorem,

$$\tau_{IPS} \;\stackrel{d}{\to}\; \mathcal{N}(0,1).$$

Thus, as long as $N \to \infty$ normality of these statistics does not require passing $T \to \infty$, a fact that is oftentimes not considered, even in theoretical work.

The performance under the stationary alternative is the topic of the next section.

Myth 1: The IPS test is more powerful than the LLC test

It has become standard to treat τ_{LLC} as a test against H_{1a} and τ_{IPS} as a test against H_{1b}. Therefore, since H_{1b} is less restrictive than H_{1a}, one might be led to believe that τ_{IPS} should dominate τ_{LLC} in terms of power, at least under the heterogeneous alternative. But this is only a myth.

Consider first the case when the slope coefficient α_i is fixed under the alternative. If H_{1a} holds, then we write

$$\frac{1}{\sqrt{NT}} \tau_{LLC} \;=\; \alpha \frac{\sqrt{\frac{1}{NT} M_{22}}}{\hat{\sigma}} + (\hat{\alpha} - \alpha) \frac{\sqrt{\frac{1}{NT} M_{22}}}{\hat{\sigma}} \;=\; O_p(1) + O_p\left(\frac{1}{\sqrt{NT}}\right) O_p(1),$$

which implies that $\tau_{LLC} = O_p(\sqrt{NT})$. Similarly, if H_{1b} holds, and assuming for simplicity that the last $N - N_1$ units are non-stationary,

$$\begin{aligned}\sqrt{\text{var}(\tau)}\,\tau_{IPS} &= \frac{1}{\sqrt{N}}\sum_{i=1}^{N}(\tau_i - E(\tau)) \\ &= \sqrt{\frac{N_1}{N}}\frac{1}{\sqrt{N_1}}\sum_{i=1}^{N_1}(\tau_i - E(\tau)) + \sqrt{1-\frac{N_1}{N}}\frac{1}{\sqrt{N-N_1}}\sum_{i=N_1+1}^{N}(\tau_i - E(\tau)) \\ &= \sqrt{\delta_1}\,O_p(\sqrt{NT}) + \sqrt{1-\delta_1}\,O_p(1),\end{aligned}$$

where we have used that

$$\frac{1}{\sqrt{T}}E(\tau_i) = \frac{\alpha_i}{\sigma}E\left(\sqrt{\frac{1}{T}M_{22i}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right) \to \frac{\alpha_i}{\sqrt{1-\rho_i^2}} \neq E(\tau)$$

as $T \to \infty$, implying

$$\frac{1}{N_1}\sum_{i=1}^{N_1}(\tau_i - E(\tau)) \xrightarrow{p} E(\tau_i) - E(\tau) = O_p(\sqrt{T})$$

so that $\frac{1}{\sqrt{N_1}}\sum_{i=1}^{N_1}(\tau_i - E(\tau)) = O_p(\sqrt{N_1 T})$, which is $O_p(\sqrt{NT})$ provided that $\delta_1 > 0$. It follows that $\tau_{IPS} = O_p(\sqrt{NT})$.

The rate of divergence is therefore the same for both tests, suggesting that their ability to reject the null should also be the same provided that N and T are large enough. Note also that the rate of divergence of τ_{IPS} is independent of the value taken by δ_1, as long as $\delta_1 > 0$. The divergence rate of this test in a panel where for example only half of the units are stationary is therefore the same as that in a panel where all units are stationary.

Consider next the case when α_i is local-to-unity,

$$H_{1c} : \alpha_i = \frac{c_i}{T\sqrt{N}}, \tag{4}$$

where $c_i < 0$ is a constant such that $\frac{1}{N}\sum_{i=1}^{N} c_i \to \bar{c}$ as $N \to \infty$. Let us assume for simplicity that $y_{i0} = 0$, then by Taylor expansion

$$\begin{aligned}\frac{1}{\sigma\sqrt{T}}y_{it} &= \frac{1}{\sigma\sqrt{T}}\sum_{j=0}^{t}\rho_i^j\varepsilon_{it-j} = \frac{1}{\sigma\sqrt{T}}\sum_{j=0}^{t}\varepsilon_{it-j} + \frac{c_i}{\sigma\sqrt{NT}}\sum_{j=1}^{t}\frac{j}{T}\varepsilon_{it-j} \\ &\Rightarrow W_i(s) + \frac{c_i}{\sqrt{N}}U_i(s)\end{aligned}$$

as $T \to \infty$, where $U_i(s) = \int_0^s W_i(r)dr$. Thus, by subsequently passing $N \to \infty$,

$$\begin{aligned}\frac{1}{\sigma^2 T\sqrt{N}}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-1}\varepsilon_{it} &\xrightarrow{d} \frac{1}{\sqrt{2}}\mathcal{N}(0,1) + \bar{c}\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}E\left(\int_0^1 U_i(s)dW_i(s)\right) \\ &\sim \frac{1}{\sqrt{2}}\mathcal{N}(0,1),\end{aligned}$$

which uses the fact that $E\left(\int_0^1 U_i(s)dW_i(s)\right) = 0$. But we also have $\frac{1}{T^2N} M_{22} \xrightarrow{p} \frac{\sigma^2}{2}$ as $N, T \to \infty$, and so we get

$$\tau_{LLC} = \frac{1}{\hat{\sigma}\sqrt{\frac{1}{NT^2}M_{22}}} \frac{1}{T\sqrt{N}} \sum_{i=1}^{N}\sum_{t=2}^{T}\left(\frac{c_i}{T\sqrt{N}} y_{it-1}^2 + y_{it-1}\varepsilon_{it}\right) \xrightarrow{d} \mathcal{N}\left(\frac{\bar{c}}{\sqrt{2}}, 1\right).$$

It is interesting to note that the denominator of the LLC statistic does not contribute to the local power of the test, which stands in sharp contrast to the DF statistic, whose local power depends on both the numerator and the denominator.

Let us now consider the local power of the IPS statistic. Using Taylor expansion and then inserting

$$\begin{aligned}
\frac{1}{\sigma^2 T}\sum_{t=2}^{T} y_{it-1}\Delta y_{it} &\Rightarrow \int_0^1 \left(W_i(r) + \frac{c_i}{\sqrt{N}}U_i(r)\right)\left(dW_i(r) + \frac{c_i}{\sqrt{N}}W_i(r)\right)dr \\
&= \int_0^1 W_i(r)dW_i(r) + \frac{c_i}{\sqrt{N}}\left(\int_0^1 U_i(r)dW_i(r) + \int_0^1 W_i(r)^2 dr\right) + O_p\left(\frac{1}{N}\right) \\
&= M_{12i}^{\circ} + \frac{c_i}{\sqrt{N}}(R_{1i} + M_{22i}^{\circ}) + O_p\left(\frac{1}{N}\right), \\
\frac{1}{\sigma^2 T^2}\sum_{t=2}^{T} y_{it-1}^2 &\Rightarrow \int_0^1 \left(W_i(r) + \frac{c_i}{\sqrt{N}}U_i(r)\right)^2 \\
&= \int_0^1 W_i(r)^2 dr + \frac{2c_i}{\sqrt{N}}\int_0^1 W_i(r)U_i(r)dr + O_p\left(\frac{1}{N}\right) \\
&= M_{22i}^{\circ} + \frac{2c_i}{\sqrt{N}} R_{2i} + O_p\left(\frac{1}{N}\right),
\end{aligned}$$

we obtain

$$\tau_i \Rightarrow \frac{M_{12i}^{\circ}}{\sqrt{M_{22i}^{\circ}}} + \frac{c_i}{\sqrt{N}}\left(\sqrt{M_{22i}^{\circ}} + \frac{R_{1i}}{\sqrt{M_{22i}^{\circ}}} - \frac{M_{12i}^{\circ}R_{2i}}{(M_{22i}^{\circ})^{3/2}}\right) + O_p\left(\frac{1}{N}\right).$$

It follows that as $N, T \to \infty$,

$$\tau_{IPS} \xrightarrow{d} \mathcal{N}(0,1) + \frac{\bar{c}}{\sqrt{\text{var}(\tau)}} E\left(\sqrt{M_{22i}^{\circ}} + \frac{R_{1i}}{\sqrt{M_{22i}^{\circ}}} - \frac{M_{12i}^{\circ}R_{2i}}{(M_{22i}^{\circ})^{3/2}}\right).$$

Using simulations where the Brownian motion $W_i(r)$ is approximated by a random walk of length $T = 1,000$ we find

$$E\left(\sqrt{M_{22i}^{\circ}} + \frac{R_{1i}}{\sqrt{M_{22i}^{\circ}}} - \frac{M_{12i}^{\circ}R_{2i}}{(M_{22i}^{\circ})^{3/2}}\right) = 0.6221 - 0.0794 + 0.0382 = 0.581.$$

Since $0.581/\sqrt{\text{var}(\tau)} = 0.581/0.985 = 0.6 < 1/\sqrt{2} = 0.707$ it follows that the local power of the IPS test is always smaller than that of the LLC test. We also see that the power only depends on the mean of c_i and not on the variance. Thus, just as in the case when α_i is treated as fixed we find that the power does not depend on the heterogeneity of the alternative.

To illustrate these findings a small simulation experiment was conducted using (1), (2) and (4) with $\varepsilon_{it} \sim \mathcal{N}(0,1)$ and $y_{i0} = 0$ to generate the data. Two specifications are considered. In the first, $c_i = c$ for all i, suggesting a completely homogenous AR parameter, while in the second, $c_i \sim U(2c, 0)$. Hence, $\text{var}(c_i) = c^2/3 > 0$ whenever $c < 0$ and so the individual AR coefficients are no longer restricted to be equal. However, the mean is still c, just as in the first specification. The empirical rejection frequencies are based on 5,000 replications and the 5% critical value.[1] The results are summarized in Table 1. We see that in agreement with the theoretical results, τ_{LLC} is uniformly more powerful than τ_{IPS}. We also see that the actual power corresponds roughly to the asymptotic power, at least for large samples and small values of c.

Table 1: Power against different local alternatives.

		$c_i = c$		$c_i \sim U(2c, 0)$	
T, N	c	LLC	IPS	LLC	IPS
20	-1	15.4	12.5	15.3	12.4
	-2	33.5	24.0	31.3	23.3
	-5	90.4	73.8	76.6	67.9
50	-1	16.6	13.1	16.2	12.8
	-2	36.6	26.8	33.9	26.1
	-5	93.7	80.4	85.0	75.9
100	-1	16.8	13.3	16.5	13.3
	-2	38.7	26.8	36.7	26.1
	-5	94.8	83.5	89.8	79.9
Asymptotic	-1	17.4	14.8	17.4	14.8
	-2	40.9	32.8	40.9	32.8
	-5	97.1	91.23	97.1	91.2

Notes: The table reports the 5% rejection frequencies when the AR parameter is set to $\alpha_i = c_i/T\sqrt{N}$.

[1] From now on all simulations will be conducted at the 5% level using 5,000 replications. Also, in order to reduce the effect of the initial condition, the last 100 observations of each cross-sectional unit will henceforth be disregarded.

3 Models with deterministic terms

Myth 2: Deterministic components should be treated as in the DF approach

In the presence of deterministic constant and trend terms, LLC and IPS suggest following the DF proposal of using least squares demeaning. One might therefore think that this is also the simplest way to handle such terms. This is only a myth.

Consider the model

$$y_{it} \;=\; \mu_i + y_{it}^s, \tag{5}$$

where the constant μ_i now represents the deterministic part of y_{it}, while y_{it}^s again represents the stochastic part. As usual, the allowance for deterministic terms of this kind makes it necessary to appropriately augment the regression in (3). Let us therefore introduce x_{it} to denote a generic vector containing all regressors other than y_{it-1} with γ_i being the associated vector of slope coefficients. In the current case with a constant this yields

$$\Delta y_{it} \;=\; \alpha_i y_{it-1} - \alpha_i \mu_i + \varepsilon_{it} \;=\; \alpha_i y_{it-1} + \gamma_i x_{it} + \varepsilon_{it}, \tag{6}$$

where $\gamma_i = -\alpha_i \mu_i$ and $x_{it} = 1$ for all i and t. The matrix of sample moments is augmented accordingly as

$$M_i \;=\; \begin{pmatrix} M_{11i} & M_{12i} & M_{13i} \\ M_{12i} & M_{22i} & M_{23i} \\ M_{13i}' & M_{23i}' & M_{33i} \end{pmatrix} \;=\; \sum_{t=2}^{T} \begin{pmatrix} (\Delta y_{it})^2 & y_{it-1}\Delta y_{it} & \Delta y_{it} x_{it}' \\ y_{it-1}\Delta y_{it} & y_{it-1}^2 & y_{it-1} x_{it}' \\ x_{it}\Delta y_{it} & x_{it} y_{it-1} & x_{it} x_{it}' \end{pmatrix}$$

with x_{it} ordered last. Moreover, since the focus here is on α_i and not on γ_i, the analysis will be carried out in two steps, where the first involves projecting Δy_{it} and y_{it-1} upon x_{it}. The second step is then to test for a unit root in the resulting projection errors, which can be written in terms of the partitions of M_i as

$$M_{abi}^p \;=\; M_{abi} - M_{a3i}\, M_{33i}^{-1}\, M_{3bi}.$$

The corresponding limiting projection error is defined as

$$M_{abi}^{\circ p} \;=\; M_{abi}^{\circ} - M_{a3i}^{\circ}\,(M_{33i}^{\circ})^{-1}\, M_{3bi}^{\circ}$$

with an obvious definition of M_{abi}°.

Also, excepting for M^p, to simplify the notation let us from now on suppress any dependence upon p. For example, we write $\hat{\sigma}^2 = \frac{1}{NT}(M_{11}^p - \hat{\alpha}\, M_{12}^p)$ and $\hat{\alpha} = M_{12}^p / M_{22}^p$, which are the same definitions as in Section 2 but with the elements of M^p in place of the corresponding elements of M.

Consider now the DF approach of using least squares demeaning, in which case

$$M_{abi}^p \;=\; M_{abi} - \frac{1}{T} M_{a3i} M_{3bi},$$

so that for example $M_{12i}^p = \sum_{t=2}^T (y_{it-1} - \overline{y}_i)\Delta y_{it}$, where $\overline{y}_i = \frac{1}{T}\sum_{t=2}^T y_{it}$ is the mean of y_{it}. The limiting version of this quantity is given by $M_{12i}^{\circ p} = \int_0^1 (W_i(s) - \overline{W}_i) dW_i(s)$, where $\overline{W}_i = \int_0^1 W_i(s) ds$. Thus, since $E(M_{12i}^{\circ p}) = -1/2$ under H_0, we have that in the sequential limit as $T \to \infty$ and then $N \to \infty$

$$\frac{1}{TN} M_{12}^p \;\Rightarrow\; \sigma^2 \frac{1}{N} \sum_{i=1}^N M_{12i}^{\circ p} \;\xrightarrow{p}\; \sigma^2 E(M_{12i}^{\circ p}) \;=\; -\frac{\sigma^2}{2}.$$

Since $\hat{\sigma}^2 \xrightarrow{p} \sigma^2$ and $\frac{1}{T^2} E(M_{22i}^p) \to \frac{\sigma^2}{6}$ we have that as $N, T \to \infty$

$$\frac{1}{\sqrt{N}} \tau_{LLC} \;=\; \frac{\frac{1}{TN} M_{12}^p}{\hat{\sigma}\sqrt{\frac{1}{T^2 N} M_{22}^p}} \;\xrightarrow{p}\; -\frac{\sqrt{6}}{2}$$

and by further use of $\frac{1}{T^2}\mathrm{var}(M_{12i}^p) \to \frac{\sigma^4}{12}$,

$$\frac{\sqrt{12N}}{\sigma^2}\left(\frac{1}{TN} M_{12}^p + \frac{\sigma^2}{2}\right) \;\xrightarrow{d}\; \mathcal{N}(0,1).$$

It follows that

$$\tau_{LLC}^c \;=\; \frac{\sqrt{2N}\left(\frac{1}{TN} M_{12}^p + \frac{\hat{\sigma}^2}{2}\right)}{\hat{\sigma}\sqrt{\frac{1}{NT^2} M_{22}^p}} \;\xrightarrow{d}\; \mathcal{N}(0,1).$$

This is the bias-adjusted LLC statistic, which has been superscripted by c to indicate that it is robust to the presence of the constant in the model. The point here is that least squares demeaning is not enough to get rid off the effect of μ_i. There is also a bias that needs to be accounted for, which complicates the testing considerably. This is the so-called Nickell bias (Nickell, 1981).

As mentioned in Section 2, as soon as one moves away from the most simple case with no deterministic components and no short-run dynamics, the statistic proposed in Levin and Lin (1992) need not be the same as the one in LLC. In the current setting Levin and Lin (1992) suggest using

$$\tau_{LL}^c \;=\; \frac{\sqrt{5}}{2}\tau_{LLC} + \sqrt{\frac{15N}{8}},$$

which is even more complicated than τ^c_{LLC}, as now it is not only the bias of the numerator but the bias of the whole test statistic that is subtracted. To appreciate the effect fo this change let us begin by expanding τ^c_{LL} as

$$\begin{aligned}\frac{2}{\sqrt{5}}\,\tau^c_{LL} &= \tau_{LLC} + \sqrt{\frac{3N}{2}} = \sqrt{N}\frac{\frac{1}{NT}M^p_{12}}{\hat{\sigma}\sqrt{\frac{1}{NT^2}M^p_{22}}} + \sqrt{N}\frac{\frac{1}{2}\,\sigma^2}{\sigma\sqrt{\frac{\sigma^2}{6}}} \\ &= \frac{\sqrt{N}\left(\frac{1}{NT}M^p_{12} + \frac{1}{2}\,\sigma^2\right)}{\hat{\sigma}\sqrt{\frac{1}{NT^2}M^p_{22}}} - \frac{1}{2}\,\sigma^2\sqrt{N}\left(\frac{1}{\hat{\sigma}\sqrt{\frac{1}{NT^2}M^p_{22}}} - \frac{1}{\sigma\sqrt{\frac{\sigma^2}{6}}}\right),\end{aligned}$$

which, by Taylor expansion of the second term, yields

$$\begin{aligned}\frac{2}{\sqrt{5}}\,\tau^c_{LL} &= \frac{\sqrt{N}\left(\frac{1}{NT}M^p_{12} + \frac{1}{2}\,\sigma^2\right)}{\sqrt{\hat{\sigma}^2\frac{1}{NT^2}M^p_{22}}} + \hat{\sigma}^2\sqrt{\frac{27}{2\sigma^{16}}}\,\sqrt{N}\left(\frac{1}{NT^2}M^p_{22} - \frac{\sigma^2}{6}\right) \\ &+ \frac{1}{NT^2}M^p_{22}\sqrt{\frac{27}{72\sigma^{16}}}\,\sqrt{N}(\hat{\sigma}^2 - \sigma^2) \\ &\xrightarrow{d} \frac{\frac{\sigma^2}{\sqrt{12}}\mathcal{N}(0,1)}{\sigma\sqrt{\frac{\sigma^2}{6}}} + \sigma^2\sqrt{\frac{27}{2\sigma^{16}}}\,\frac{\sigma^2}{\sqrt{45}}\mathcal{N}(0,1),\end{aligned}$$

where we have used that $\frac{1}{T^2}\,\mathrm{var}(M^p_{22i}) \to \frac{\sigma^4}{45}$ and $\sqrt{N}(\hat{\sigma}^2 - \sigma^2) = o_p(1)$, see Lemma 2 of Moon and Phillips (2004). It follows that

$$\frac{2}{\sqrt{5}}\,\tau^c_{LL} \xrightarrow{d} \left(\frac{1}{\sqrt{2}} + \sqrt{\frac{3}{10}}\right)\mathcal{N}(0,1) \sim \frac{2}{\sqrt{5}}\mathcal{N}(0,1),$$

or $\tau^c_{LL} \xrightarrow{d} \mathcal{N}(0,1)$.

Thus, although the end result is the same as for τ^c_{LLC}, the route to normality is more complicated than for τ^c_{LL}, and involves additional approximations, which is suggestive of poor small-sample properties. On the other hand, the bias-adjustment of LLC requires estimation of σ^2, which obviously increases the variability of their test.

The relationship between the two statistics is easily seen by noting that

$$\begin{aligned}\frac{2}{\sqrt{5}}\,\tau^c_{LL} &= \tau_{LLC} + \sqrt{\frac{3N}{2}} = \tau_{LLC} + \frac{1}{2}\,\sqrt{N}\,\frac{\sigma}{\sqrt{\frac{\sigma^2}{6}}} \\ &= \tau_{LLC} + \frac{1}{2}\,\sqrt{N}\lim_{N,T\to\infty}\frac{\hat{\sigma}}{\sqrt{\frac{1}{NT^2}M^p_{22}}} = \tau_{LLC} + \frac{1}{2}\,\sqrt{N}\lim_{N,T\to\infty} T\sqrt{N}\frac{\hat{\sigma}}{\sqrt{M^p_{22}}},\end{aligned}$$

which is asymptotically equivalent to

$$\tau^c_{LLC} = \sqrt{2}\,\tau_{LLC} + \frac{NT}{\sqrt{2}}\frac{\hat{\sigma}}{\sqrt{M^p_{22}}}.$$

However, the demeaning not only complicates the route to normality but also impact the local power of the tests. Consider τ_{LLC}^c. From Moon and Perron (2008) we have that under H_{1c},

$$\frac{1}{\sigma\sqrt{T}}(y_{it} - \overline{y}_i) \;\Rightarrow\; W_i(s) - \overline{W}_i + \frac{c_i}{\sqrt{N}}(U_i(s) - \overline{U}_i) + O_p\left(\frac{1}{N^{3/4}}\right)$$

as $T \to \infty$, which implies

$$\frac{1}{\sigma^2 T} M_{12i}^p \;\Rightarrow\; M_{12i}^{\circ p} + \frac{c_i}{\sqrt{N}}\left(M_{22i}^{\circ p} + \int_0^1 (U_i(r) - \overline{U}_i) dW_i(r)\right) + O_p\left(\frac{1}{N^{3/4}}\right).$$

Using $E(M_{12i}^{\circ p}) = \sigma^2/2$, $E(M_{22i}^{\circ p}) = \sigma^2/6$, $\text{var}(M_{12i}^{\circ p}) = \sigma^2/12$ and

$$E\left(\int_0^1 (U_i(r) - \overline{U}_i) dW_i(r)\right) \;=\; -E(W_i(1)\overline{U}_i) \;=\; -\frac{1}{6}$$

it is possible to show that as $N, T \to \infty$

$$\begin{aligned} \frac{\sqrt{12N}}{\sigma^2}\left(\frac{1}{NT} M_{12}^p + \frac{\sigma^2}{2}\right) &\;\xrightarrow{d}\; \mathcal{N}(0,1) + \sqrt{12}\,\overline{c}\,E\left(M_{22i}^{\circ p} + \int_0^1 (U_i(r) - \overline{U}_i) dW_i(r)\right) \\ &\;\sim\; \mathcal{N}(0,1). \end{aligned}$$

Hence, under the typical sequence of local alternatives given by (4) the limiting distribution of the numerator of τ_{LLC}^c does not depend on c_i. For the denominator we have

$$\frac{1}{\sigma^2 T^2} M_{22i}^p \;\Rightarrow\; M_{22i}^{\circ p} + \frac{2c_i}{\sqrt{N}} \int_0^1 (W_i(r) - \overline{W}_i)(U_i(r) - \overline{U}_i) dr + +O_p\left(\frac{1}{N^{3/4}}\right),$$

suggesting that as $T \to \infty$ and then $T \to \infty$

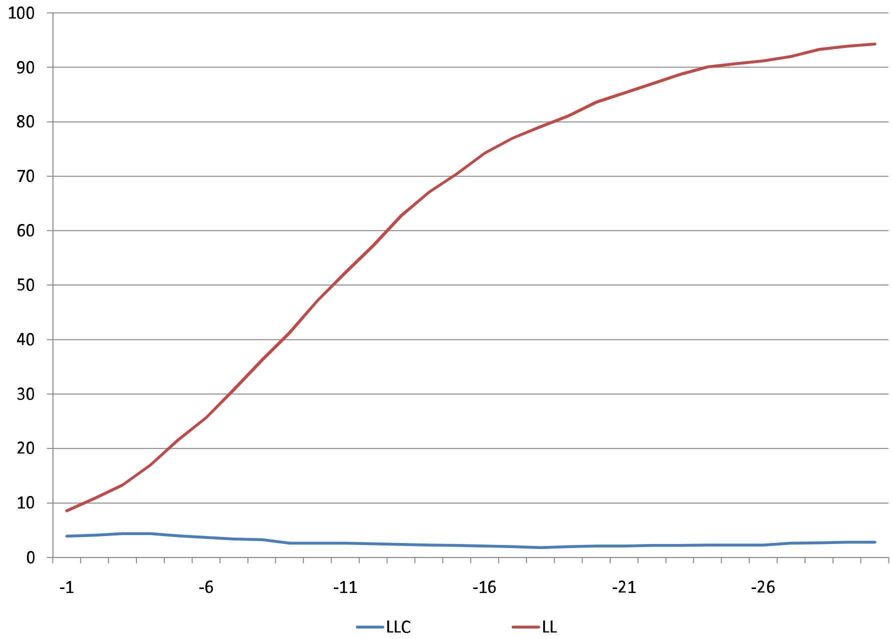
$$\frac{1}{NT^2} M_{22}^p \;\Rightarrow\; \sigma^2 \frac{1}{N}\sum_{i=1}^N M_{22i}^{\circ p} + O_p\left(\frac{1}{\sqrt{N}}\right) \;\xrightarrow{p}\; \sigma^2 E(M_{22i}^{\circ p}) \;=\; \frac{\sigma^2}{6},$$

from which it follows that

$$\tau_{LLC}^c \;=\; \frac{\sqrt{2N}\left(\frac{1}{TN} M_{12}^p + \frac{\hat{\sigma}^2}{2}\right)}{\hat{\sigma}\sqrt{\frac{1}{NT^2} M_{22}^p}} \;\xrightarrow{d}\; \mathcal{N}(0,1).$$

In other words, unlike τ_{LL}^c, τ_{LLC}^c does not have any power against H_{1c}. This is illustrated in Figure 1, which plots the local power as a function of c when the data are generated from (1), (2) and (4) with $c_i \sim U(2c, 0)$. As in Table 1, the results are based on 5,000 replications and the 5% critical value. Note in particular how the power function of τ_{LL}^c is strictly increasing in c, while that of τ_{LLC}^c is flat. As it turns out this loss of power can be easily explained, an issue that we will discuss to some extent in Section 4.

Figure 1: Local power of τ_{LL}^c and τ_{LLC}^c.



The point here is that these complications are all due to the fact that the constant is removed by least squares demeaning. Thus, in order to avoid bias and corrections thereof, one needs to consider alternatives to least squares demeaning. For example, Breitung and Meyer (1994) suggest using the initial value y_{i0} as an estimator of γ_i, and to test for a unit root in a regression of Δy_{it} on $y_{it-1}^* = y_{it-1} - y_{i0}$. To see how this is going to affect the results, note that

$$E(\Delta y_{it} y_{it-1}^*) = E\big(\Delta y_{it}\,(y_{it-1} - y_{i0})\big) = E\left(\varepsilon_{it} \sum_{s=1}^{t-1} \varepsilon_{is}\right) = 0.$$

In other words, using y_{i0} as an estimator of γ_i removes the bias. In fact, it is not difficult to show that as $N, T \to \infty$

$$\tau_{BM}^c = \frac{\sum_{i=1}^N \sum_{t=2}^T y_{it-1}^* \Delta y_{it}}{\hat{\sigma}\sqrt{\sum_{i=1}^N \sum_{t=2}^T (y_{it-1}^*)^2}} \overset{d}{\to} \mathcal{N}(0,1),$$

where τ_{BM}^c is the Breitung and Meyer (1994) statistic.

Interestingly, as pointed out by Phillips and Schmidt (1992), y_{i0} is also the maximum likelihood estimator of γ_i under H_0, which has been shown to lead to significant power gains when compared to least squares demeaning, see Madsen (2003). In fact, it is not difficult to see that under H_{1c},

$$\tau_{BM}^{c} \overset{d}{\to} \mathcal{N}\left(\frac{\bar{c}}{\sqrt{2}}, 1\right)$$

as N, $T \to \infty$, which is the same results we obtained earlier for the LLC statistic in the model without any deterministic terms.

To examine the extent of these gains in small samples Table 2 reports some results based on data generated from (1), (2) and (4) with $c_i \sim U(2c, 0)$. Consistent with the results of Madsen (2003) we see that the tests based on removing the initial condition are almost uniformly more powerful than those based on least squares demeaning. We also see that this increase in power comes at no cost in terms of size accuracy.

Table 2: Size and local power for different demeaning procedures.

			LLC		IPS	
c	N	T	LS	ML	LS	ML
0	10	50	7.1	7.1	7.3	5.8
	20	50	6.8	6.9	7.4	6.3
	10	100	6.4	7.5	4.8	5.4
	20	100	6.6	7.0	5.3	5.8
−1	10	50	10.1	11.9	9.4	8.5
	20	50	9.3	12.9	9.7	10.3
	10	100	8.6	13.5	6.7	10.0
	20	100	9.6	13.2	8.2	11.2
−2	10	50	13.2	18.5	10.5	12.5
	20	50	12.5	20.1	11.6	14.4
	10	100	10.9	18.0	7.0	14.4
	20	100	11.7	19.9	10.0	16.2
−5	10	50	25.9	41.0	19.8	31.2
	20	50	23.7	46.1	20.9	37.3
	10	100	21.6	39.8	15.3	33.3
	20	100	24.0	43.3	17.3	36.8

Notes: The table reports the 5% rejection frequencies when the AR parameter is set to $\alpha_i = c_i/T\sqrt{N}$, where $c_i \sim U(2c, 0)$. LS and ML refer to demeaning by least squares and maximum likelihood, respectively, where the latter is based on removing the first observation from y_{it}.

Another possibility is to demean y_{it-1} recursively as $y_{it-1} - \frac{1}{t-1}\sum_{s=0}^{t-1} y_{is}$, which can be used instead of y^*_{it-1} to produce yet another unbiased and standard normally distributed test statistic.

These alternative demeaning approaches can be seen as special cases of a more general class of test statistics. In particular, letting $\Delta y_i = \begin{pmatrix} \Delta y_{i2}, & \dots, & \Delta y_{iT} \end{pmatrix}'$ and $y_{i,-1} = \begin{pmatrix} y_{i0}, & \dots, & y_{iT-1} \end{pmatrix}'$, these statistics can be written as

$$\frac{\sum_{i=1}^N (\Delta y_i)' C y_{i,-1}}{\hat{\sigma}\sqrt{\sum_{i=1}^N y'_{i,-1} C' C y_{i,-1}}}.$$

The matrix C has the property that $C\,\iota_T = 0$, where ι_T is a vector of ones. Therefore, pre-multiplying $y_{i,-1}$ by C eliminates the individual specific constant. The statistic has expectation zero if

$$E\big((\Delta y_i)' C y_{i,-1}\big) \;=\; \sigma^2\, tr(CD) \;=\; 0,$$

where D is a matrix with elements $d_{jk} = 1$ if $j < k$ and $d_{jk} = 0$ for $j \geq k$. Note that in the case of least squares demeaning, $C = I_T - \frac{1}{T}\iota_T \iota_T'$, where I_T is the identity matrix. Since in this case $tr(CD) \neq 0$, bias correction is needed.

The same principle can be used to construct bias-corrected statistics in models with trends, an issue to be discussed in the next section.

Fact 2: Incidental trends reduces the local power of the LLC test

Suppose now that instead of (5) we have

$$y_{it} \;=\; \mu_i + \beta_i\, t + y^s_{it}, \tag{7}$$

where $\beta_i\, t$ is a unit specific trend term, giving

$$\Delta y_{it} \;=\; -\alpha_i \mu_i + (\alpha_i + 1)\beta_i - \alpha_i \beta_i\, t + \alpha_i y_{it-1} + \varepsilon_{it} \;=\; \alpha_i y_{it-1} + \gamma_i' x_{it} + \varepsilon_{it}$$

with $x_{it} = \begin{pmatrix} 1, & t \end{pmatrix}'$.

The incidental trends problem refers to the need of having to estimate the trend coefficient β_i, whose number goes to infinity as $N \to \infty$, which reduces the discriminatory power against H_0, see Moon and Phillips (1999). In particular, as we will now demonstrate the presence of trends even has an order effect on the neighborhoods around the unit root null for which asymptotic power is non-negligible.

As Moon *et al.* (2007) show in the case with incidental trends the LLC statistic is asymptotically equivalent to

$$\tau_{LLC}^t = \frac{193}{112}\tau_{LLC} + \sqrt{\frac{252}{772}}\frac{10}{T}\frac{\sqrt{M_{22}^p}}{\hat{\sigma}},$$

where the superscript t indicates invariance with respect to the trend, while τ_{LLC} is now the LLC statistic based on the detrended data. Moon and Perron (2004) consider another statistic, which in the present setting may be written as

$$\tau_{MP}^t = \tau_{LLC} + \frac{NT}{2}\frac{\hat{\sigma}}{\sqrt{M_{22}^p}}.$$

It follows that

$$\sqrt{\frac{193}{112}}\tau_{LLC}^t = \tau_{MP}^t + \frac{15}{2T}\frac{\left(M_{22}^p - \frac{1}{15}\hat{\sigma}^2\right)}{\hat{\sigma}\sqrt{M_{22}^p}},$$

suggesting that τ_{LLC}^t will inherit some of the asymptotic properties of τ_{MP}^t. In particular, from Theorem 4 of Moon and Perron (2004) we know that τ_{MP}^t has power within $\frac{1}{N^{1/4}T}$ neighborhoods of H_0, but not for any higher powers of N and T. In particular, τ_{MP}^t has no power against H_{1c} when the neighborhood is of order $\frac{1}{T\sqrt{N}}$. The above relationship imply that τ_{LLC}^t has the same property. Thus, just as in the case of an intercept, we see that the presence of the trend leads to a loss of power. This is illustrated in Table 3, which plots the local power of the LLC and IPS tests for some different values of c when α_i is generated according to (4) with $c_i \sim U(2c, 0)$. In accordance with the theoretical results we see that the power can be very low and practically nonexisting in many cases if there is a trend in the model.

In view of the previous myth one might think that this loss of power is due to the fact that the detrending is carried out using least squares. However, this is not true. Take as an example the study of Breitung (2000), who proposes a generalized version of the demeaning by initial value procedure discussed in the previous section. Specifically, using y_{i0} and $\frac{1}{T}\sum_{t=2}^T \Delta y_{it} = \frac{1}{T}(y_{iT} - y_{i0})$ as estimators of the constant and trend, respectively, Breitung (2000) proposes replacing (8) with a regression of Δy_{it}^* on y_{it-1}^*, where $y_{it}^* = y_{it} - y_{i0} - \frac{1}{T}(y_{iT} - y_{i0})\,t$ and

$$\Delta y_{it}^* = s_t\left(\Delta y_{it} - \frac{1}{T-t}(y_{iT} - y_{it})\right)$$

Table 3: Local power in the presence of incidental trends.

c	N	T	LLC	IPS
-1	10	50	9.9	10.9
	20	50	10.6	12.2
	10	100	6.9	7.7
	20	100	8.2	8.4
-4	10	50	13.4	12.4
	20	50	12.6	14.3
	10	100	10.2	9.4
	20	100	10.0	10.4
-8	10	50	22.8	22.1
	20	50	20.0	20.5
	10	100	16.9	14.6
	20	100	14.9	14.9

Notes: The table reports the 5% rejection frequencies when the AR parameter is set to $\alpha_i = c_i/T\sqrt{N}$, where $c_i \sim U(2c, 0)$.

with $s_t^2 = (T-t)/(T-t+1)$. The effect of this is easily seen by noting that

$$\begin{aligned} E(\Delta y_{it}^* y_{it-1}^*) &= s_t\, E\left(\left(\Delta y_{it} - \frac{1}{T-t}(y_{iT} - y_{it})\right)\left(y_{it} - y_{i0} - \frac{1}{T}(y_{iT} - y_{i0})\right)\right) \\ &= s_t\, E\left(\left(\Delta y_{it}^s - \frac{1}{T-t}(y_{iT}^s - y_{it}^s)\right)\left(y_{it}^s - y_{i0}^s - \frac{1}{T}(y_{iT}^s - y_{i0}^s)\right)\right) \\ &= s_t\, E\left(\left(\varepsilon_{it} - \frac{1}{T-t}(y_{iT}^s - y_{it}^s)\right)\left(y_{it-1}^s - \frac{t-1}{T}y_{iT}^s\right)\right) \\ &= s_t\left(\frac{t-1}{T}\sigma^2 - \frac{(t-1)(T-t)}{(T-t)T}\sigma^2\right) \;=\; 0, \end{aligned}$$

showing that the bias has been successfully eliminated.

However, as Moon *et al.* (2006) show, just as with τ_{MP}^t and τ_{LLC}^t, the Breitung (2000) test has no power in neighborhoods that shrinks to zero at a faster rate than $\frac{1}{N^{1/4}T}$. The reduced power effect in the presence of trends is therefore not specific to τ_{MP}^t and τ_{LLC}^t but is a general property of this type of tests. In fact, as Ploberger and Phillips (2002) show, the panel unit root test that maximizes the average local power has significant power in local neighborhoods that shrink at the same rate, $\frac{1}{N^{1/4}T}$.

Myth 3: The initial condition does not affect the asymptotic properties of the tests

The power of panel unit root tests is usually evaluated while assuming that all N units are initiated at zero. Although this is a convenient assumption that simplifies the theoretical considerations, it is very unrealistic and, as we will see, by no means innocuous. Suppose for example that y_{it} is generated according to (5) with a constant and where y_{it}^s is as in (2). But suppose now that instead of setting y_{i0}^s to zero, we set

$$y_{i0}^s = \frac{\sigma}{\sqrt{1-\rho_i^2}}\eta_i$$

where η_i is independent and identically distributed with mean $\overline{\eta}$ and variance σ_η^2. Similar to what we had before when $y_{i0}^s = 0$, Harris *et al.* (2009) show that under H_{1c}, as $T \to \infty$

$$\begin{aligned}\frac{1}{\sigma\sqrt{T}}(y_{it} - \overline{y}_i) \;\Rightarrow\; & \frac{\eta_i}{N^{1/4}}\left(r - \frac{1}{2}\right)\sqrt{\frac{-c_i}{2}} + W_i(s) - \overline{W}_i + \frac{c_i}{\sqrt{N}}(U_i(s) - \overline{U}_i) \\ & + O_p\left(\frac{1}{N^{3/4}}\right),\end{aligned}$$

implying

$$\begin{aligned}\frac{1}{\sigma^2 T}M_{12i}^p \;\Rightarrow\; & M_{12i}^{op} + \frac{c_i}{\sqrt{N}}\left(M_{22i}^{op} + \int_0^1 (U_i(r) - \overline{U}_i)dW_i(r)\right) \\ & - \frac{\eta_i}{N^{1/4}}\sqrt{\frac{-c_i}{2}}\int_0^1\left(r - \frac{1}{2}\right)dW_i(r) + O_p\left(\frac{1}{N^{3/4}}\right).\end{aligned}$$

It follows that as $N, T \to \infty$

$$\begin{aligned}\frac{\sqrt{12N}}{\sigma^2}\left(\frac{1}{NT}M_{12}^p + \frac{\sigma^2}{2}\right) \;\xrightarrow{d}\; & \mathcal{N}(0,1) + \sqrt{12}\,\overline{c}\,E\left(M_{22i}^{op} + \int_0^1 (U_i(r) - \overline{U}_i)dW_i(r)\right) \\ & - \lim_{N\to\infty}\frac{1}{N^{3/4}}\sum_{i=1}^N \eta_i\sqrt{\frac{-c_i}{2}}\int_0^1\left(r - \frac{1}{2}\right)dW_i(r).\end{aligned}$$

But $E\big(\eta_i\sqrt{-c_i/2}\int_0^1(r - 1/2)dW_i(r)\big) = 0$, suggesting that the last term on the right-hand side is $O_p(1/N^{1/4})$. Thus, since the second term is zero,

$$\frac{\sqrt{12N}}{\sigma^2}\left(\frac{1}{NT}M_{12}^p + \frac{\sigma^2}{2}\right) \xrightarrow{d} \mathcal{N}(0,1),$$

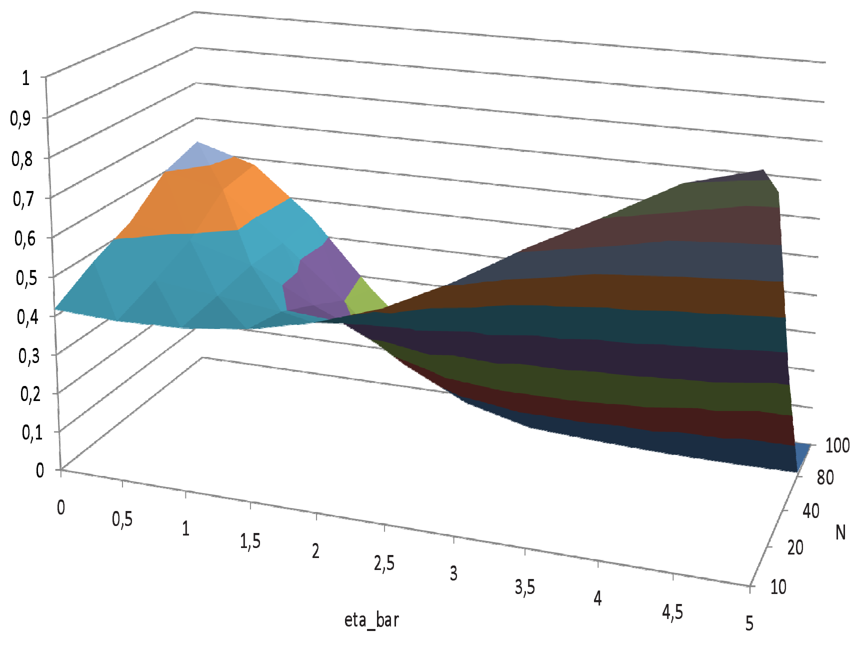
which is the same result as the one we obtained when $y_{i0}^s = 0$. By further using $\frac{1}{T^2N}M_{22}^p \xrightarrow{p} \sigma^2/6$ it follows that $\tau_{LLC}^c \xrightarrow{d} \mathcal{N}(0,1)$. Thus, just as before the asymptotic distribution of τ_{LLC}^c

is independent of both $\overline{c}$ and y^s_{i0}. However, this is not the case with the IPS test. In fact, as Harris *et al.* (2009) show,

$$\tau^c_{IPS} \xrightarrow{d} \mathcal{N}(0,1) + \overline{c}\left(0.282 - 0.135(\overline{\eta}^2 + \sigma^2_\eta)\right),$$

which shows that the local power of the IPS test is decreasing in $\overline{\eta}^2$ and σ^2_η. In particular, note that if the initial condition is large enough so that $0.282 > 0.135(\overline{\eta}^2 + \sigma^2_\eta)$, then this test is no longer consistent. This is illustrated in Figures 2 and 3, which plot the local power of the IPS test for different combinations of N and $\overline{\eta}$ when $\sigma^2_\eta = 0$ and α_i is generated as in (4) with $c_i = -10$ for all i. We see that if there is a constant present then the power is decreasing in both N and $\overline{\eta}$, while if there is no deterministic component then the power is almost perfect.

Figure 2: Local power of τ^c_{IPS} for different initial values.

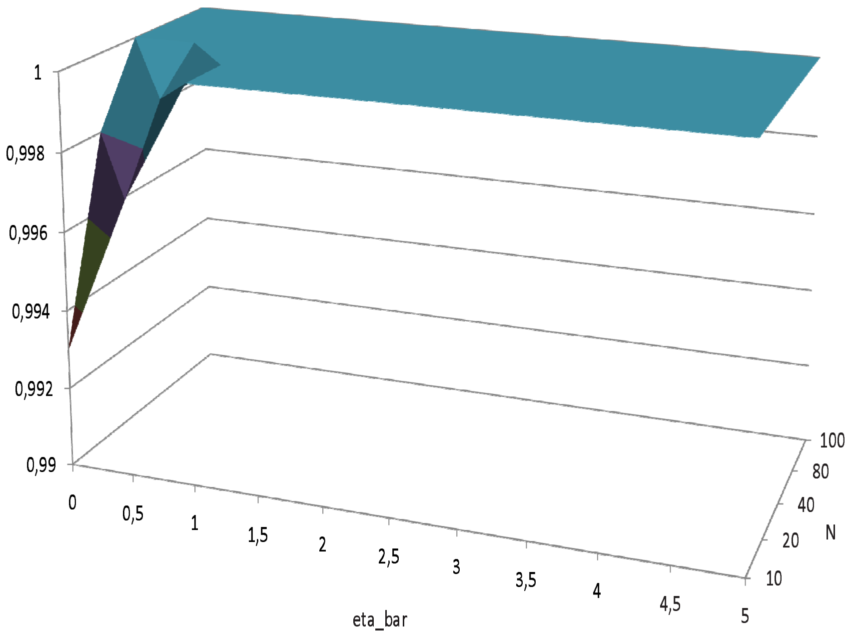


4 Models with short-run dynamics

Myth 4: Lag augmentation removes the effects of serial correlation

Suppose that (1) holds so that y_{it} is purely stochastic, but that the error ε_{it} in (2) is no longer independent across t. In particular, suppose that ε_{it} follows a stationary and invertible AR

Figure 3: Local power of τ_{IPS} for different initial values.



process of order p,

$$\phi(L)\varepsilon_{it} \;=\; \left(1-\sum_{j=1}^{p}\phi_j L^j\right)\varepsilon_{it} \;=\; \varepsilon_{it}-\sum_{j=1}^{p}\phi_j\varepsilon_{it-j} \;=\; e_{it}, \tag{8}$$

where L is the lag operator and e_{it} is a mean zero error that has variance σ^2 for all i but is otherwise independent across both i and t. As with the homoskedasticity of e_{it} the assumption of homogenous lag coefficients is not necessary but is made here in order to simplify the presentation. In particular, it means that the long-run variance of ε_{it},

$$\omega^2 \;=\; \frac{\sigma^2}{\phi(1)^2},$$

does not have to carry an i index.

Under H_0, (1), (2) and (8) can be combined to obtain the following augmented DF (ADF) regression:

$$\Delta y_{it} \;=\; \alpha_i y_{it-1}+\sum_{j=1}^{p}\phi_j\Delta y_{it-j}+e_{it} \;=\; \alpha_i y_{it-1}+\gamma' x_{it}+e_{it}, \tag{9}$$

where $x_{it} = (\;\Delta y_{it-1},\;\ldots,\;\Delta y_{it-p}\;)'$ is now the vector of lagged differences with $\gamma = (\;\phi_1,\;\ldots,\;\phi_p\;)'$ being the associated vector of lag coefficients. This gives rise to the ADF test statistic,

$$\tau_i = \frac{M^p_{12i}}{\hat{\sigma}_i \sqrt{M^p_{22i}}} = \hat{\alpha}_i \, \frac{\sqrt{M^p_{22i}}}{\hat{\sigma}_i},$$

where $\hat{\sigma}^2_i = \frac{1}{T}(M^p_{11i} - \hat{\alpha}_i \, M^p_{12i})$ and $\hat{\alpha}_i = M^p_{12i}/M^p_{22i}$, which again suppress the dependence upon p. Note also that in this setup M^p_{abi} takes the projection onto the lags of Δy_{it} rather than onto a vector of deterministic components as in Section 3.

Under H_0,

$$\begin{aligned}
\frac{1}{T} M^p_{12i} &= \frac{1}{T} M_{12i} - \frac{1}{T} M_{13i} \, M^{-1}_{33i} \, M_{32i} = \frac{1}{T} M_{12i} + \frac{1}{T} O_p(\sqrt{T}\,) O_p\left(\frac{1}{T}\right) O_p(T) \\
&= \frac{1}{T} M_{12i} + O_p\left(\frac{1}{\sqrt{T}}\right), \\
\frac{1}{T^2} M^p_{22i} &= \frac{1}{T^2} M_{22i} - \frac{1}{T^2} M_{23i} \, M^{-1}_{33i} \, M_{32i} = \frac{1}{T^2} M_{22i} - \frac{1}{T^2} O_p(T) O_p\left(\frac{1}{T}\right) O_p(T) \\
&= \frac{1}{T^2} M_{22i} + O_p\left(\frac{1}{T}\right).
\end{aligned}$$

These results, together with $\frac{1}{T} M^p_{11i} \xrightarrow{p} \sigma^2$ and

$$\begin{pmatrix} \frac{1}{T} \, M_{12i} \\ \frac{1}{T^2} \, M_{22i} \end{pmatrix} \Rightarrow \begin{pmatrix} \sigma\omega \, M^{\circ}_{12i} \\ \omega^2 \, M^{\circ}_{22i} \end{pmatrix}$$

imply that

$$\tau_i = \frac{M^p_{12i}}{\hat{\sigma}_i \sqrt{M^p_{22i}}} \Rightarrow \frac{M^{\circ}_{12i}}{\sqrt{M^{\circ}_{22i}}}.$$

Thus, the asymptotic distribution of τ_i is not affected by the presence of short-run dynamics, suggesting that the distribution of the IPS statistic should be unaffected too, see Section 4 of IPS. In other words, with this test lag augmentation successfully removes the short-run dynamics of the panel. This is also true for the LLC statistic if the model does not include deterministic terms. To see this note that

$$\frac{1}{T\sqrt{N}} M^p_{12} \xrightarrow{d} \frac{\sigma\,\omega}{\sqrt{2}} \mathcal{N}(0,1)$$

as N, $T \to \infty$. Furthermore,

$$\frac{1}{NT^2} M^p_{22} \xrightarrow{p} \frac{\omega^2}{2},$$

from which it follows that

$$\tau_{LLC} = \frac{M^p_{12}}{\hat{\sigma}\sqrt{M^p_{22}}} \xrightarrow{d} \mathcal{N}(0,1).$$

Thus, lag augmentation removes the effect of the short-run dynamics also for the LLC statistic. However, the situation changes dramatically if the model includes a constant or a linear time trend. Consider for example the case with short-run dynamics and a constant, in which we let $x_{it} = (\ 1,\ \ \Delta y_{it-1},\ \ \ldots,\ \ \Delta y_{it-p}\ )'$ and re-define M_{12i}^p and M_{22i}^p accordingly. This yields

$$\lim_{N,T\to\infty} E\left(\frac{1}{TN} M_{12}^p\right) = \sigma\,\omega \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} E(M_{12i}^{\circ}) \ \to\ -\frac{\sigma\,\omega}{2}.$$

It follows that in this case lag augmentation does not remove short-run parameters from the mean of the statistic. To cope with this problem LLC propose a bias and serial correlation corrected version of τ_{LLC}, which we again denote by τ_{LLC}^c. The problem is that since the mean of τ_{LLC} depends on both σ^2 and ω^2, for the bias-correction to work these parameters have to be estimated consistently, an issue that will be considered in more detail below. It follows that in the presence of deterministic terms lag augmentation alone is not enough to remove the short-run parameters from the asymptotic distribution of the LLC statistic.

For the estimation of ω^2 LLC propose using

$$\hat{\omega}_i^2 = \frac{1}{T}\sum_{t=2}^{T}(\Delta y_{it})^2 + \frac{2}{T}\sum_{j=1}^{q-1}\left(1-\frac{j}{q}\right)\sum_{t=j+1}^{T}\Delta y_{it}\Delta y_{it-j},$$

which is the conventional Newey and West (1994) long-run variance estimator. It is important to note that by using Δy_{it} this estimator is in fact imposing H_0. Thus, if H_0 holds then we have from Andrews (1991) that $\hat{\omega}_i^2 \xrightarrow{p} \omega^2$ as $T\to\infty$ with $q\to\infty$ and $\frac{q}{T}\to 0$, suggesting that

$$\hat{\omega} = \frac{1}{N}\sum_{i=1}^{N}\hat{\omega}_i \ \xrightarrow{p}\ \omega.$$

This indicates that the following bias-corrected statistic can be used

$$\tau_{LLC}^c = \sqrt{2}\,\tau_{LLC} + \frac{NT}{\sqrt{2}}\frac{\hat{\omega}}{\sqrt{M_{22}^p}},$$

whose asymptotic distribution can be obtained by writing

$$\tau_{LLC}^c = \frac{\sqrt{2N}\left(\frac{1}{TN}M_{12}^p + \frac{1}{2}\hat{\sigma}\,\hat{\omega}\right)}{\hat{\sigma}\sqrt{\frac{1}{NT^2}M_{22}^p}} \ \xrightarrow{d}\ \frac{\frac{\sigma\omega}{\sqrt{6}}\mathcal{N}(0,1)}{\sigma\sqrt{\frac{\omega^2}{6}}} \ \sim\ \mathcal{N}(0,1),$$

which is the result presented in Theorem 5 of LLC. However, although theoretically not an issue as long as $q\to\infty$ and $\frac{q}{T}\to 0$, in practice the optimal q to use in a given sample is never known, which of course adds to the variability of the statistic. Then there is also the problem that $\hat{\omega}^2$ tends to zero under H_{1a}, an issue that we will discuss more in the next section.

To sidestep these difficulties, Breitung and Das (2005) propose to pre-whiten the variables. Their idea is as follows. Under H_0 we have

$$\Delta y_{it} \;=\; \sum_{j=1}^{p} \phi_j \Delta y_{it-j} + e_{it}, \tag{10}$$

or, in terms of levels,

$$y_{it} \;=\; \sum_{s=2}^{t} \Delta y_{is} \;=\; \sum_{j=1}^{p} \phi_j \sum_{s=2}^{t} \Delta y_{is-j} + \sum_{s=2}^{t} e_{is} \;=\; \sum_{j=1}^{p} \phi_j y_{it-j} + y_{it}^{s},$$

where y_{it}^s is as in (2) but with H_0 imposed. It is a random walk with serially uncorrelated increments e_{it}. Thus, in contrast to $\frac{1}{\sqrt{T}}\, y_{it}$, whose long-run variance is given by ω^2, the long-run variance of $\frac{1}{\sqrt{T}}\, y_{it}^s$ is just σ^2. For the estimation of the lag coefficients ϕ_j, Breitung and Das (2005) recommend using the above regression in first differences with H_0 imposed. For a fixed p this yields

$$\frac{1}{\sqrt{T}}\, y_{it}^{*} \;=\; \frac{1}{\sqrt{T}}\, y_{it}^{s} - \sum_{j=1}^{p} (\hat{\phi}_{ij} - \phi_j)\, \frac{1}{\sqrt{T}}\, y_{it-j} \;=\; \frac{1}{\sqrt{T}}\, y_{it}^{s} + O_p\left(\frac{1}{\sqrt{T}}\right) O_p(1),$$

where $\hat{\phi}_{ij}$ is the least squares estimate of ϕ_j in (10), which means that $\sqrt{T}(\hat{\phi}_{ij} - \phi_j) = O_p(1)$. Similarly,

$$\Delta y_{it}^{*} \;=\; e_{it} - \sum_{j=1}^{p} (\hat{\phi}_{ij} - \phi_j) \Delta y_{it-j} \;=\; e_{it} + O_p\left(\frac{1}{\sqrt{T}}\right).$$

Thus, replacing Δy_{it} and y_{it} by Δy_{it}^* and y_{it}^*, respectively, eliminates the effects of the serial correlation without requiring any estimation of ω^2.

In Table 4 we compare the size accuracy of the IPS and LLC tests using both least squares lag augmentation and pre-whitening. The data are generated from (1), (2) and (8) with $p = 1$, which makes ϕ_1, the first-order AR coefficient, an interesting nuisance parameter to study. For the choice of lag length we consider three alternatives. The first is to ignore the serial correlation and to set $p = 0$, while the second is to set p equal to its true value. The third alternative is to chose p in a data-dependent fashion by using the Schwarz Bayesian information criterion with a maximum of five lags.

The first thing to notice is the size distortions that result from ignoring the serial correlation, especially when $\phi_1 = -0.5$, and the effectiveness by which they are removed in the two correction procedures. Note also that there are basically no differences in the results depending on whether p is treated as known or not.

Table 4: Size for different corrections for short-run dynamics.

		$\phi_1 = 0.5$				$\phi_1 = -0.5$			
		LLC		IPS		LLC		IPS	
N	T	Aug	Pre	Aug	Pre	Aug	Pre	Aug	Pre
				No lags					
10	20	7.5	7.5	3.8	3.8	13.4	13.4	21.7	21.7
20	20	5.2	5.2	3.1	3.1	22.4	22.4	31.7	31.7
40	20	2.7	2.7	0.6	0.6	45.1	45.1	54.4	54.4
10	50	4.8	4.8	1.7	1.7	29.2	29.2	42.0	42.0
20	50	2.8	2.8	0.4	0.4	50.0	50.0	67.5	67.5
40	50	0.7	0.7	0.1	0.1	82.9	82.9	90.6	90.6
				The true number of lags					
10	20	8.9	7.1	5.6	3.5	8.4	7.6	4.5	4.0
20	20	8.9	5.8	3.5	2.7	7.6	7.2	3.2	3.0
40	20	8.2	6.1	2.0	0.9	6.0	5.7	1.7	1.3
10	50	7.5	6.7	3.6	2.6	6.5	6.0	3.5	3.4
20	50	6.9	5.5	3.5	2.8	6.2	5.8	3.6	3.5
40	50	6.1	5.3	2.5	2.0	5.2	5.2	2.8	2.7
				The Schwarz Bayesian information criterion					
10	20	10.0	7.3	6.0	3.2	8.1	8.0	4.8	4.2
20	20	9.3	6.8	4.3	2.8	7.9	7.1	3.5	3.0
40	20	8.0	5.9	2.1	0.9	6.0	5.3	2.1	1.5
10	50	7.5	6.7	3.6	2.6	6.4	6.1	3.4	3.4
20	50	6.9	5.6	3.6	2.8	6.3	5.8	3.6	3.4
40	50	5.9	5.2	2.5	1.7	5.2	5.3	2.9	2.7

Notes: The table reports the 5% rejection frequencies under H_0. ϕ_1 refers to the first-order AR serial correlation parameter. Aug and Pre refer to least squares augmentation and pre-whitening, respectively.

Fact 3: The long-run variance estimator of LLC is inconsistent under the alternative hypothesis

Provided that $q \to \infty$ such that $\frac{q}{T} \to 0$, then we have that under H_0 $\sqrt{T}(\hat{\omega}_i^2 - \omega^2) = O_p(1)$, which via Taylor expansion gives

$$\hat{\omega} = \frac{1}{N}\sum_{i=1}^{N}\hat{\omega}_i = \frac{1}{N}\sum_{i=1}^{N}\omega + O_p\left(\frac{1}{\sqrt{T}}\right) \xrightarrow{p} \omega$$

as N, $T \to \infty$. Thus, provided that H_0 holds, $\hat{\omega}$ is consistent for ω, which as we have seen is a requirement for τ_{LLC}^c to be asymptotically normal. The problem is that if H_0 is false, because y_{it} is stationary, Δy_{it} is over-differentiated with no variance at zero frequency. Thus, in contrast to what happens under H_0, in this case $\hat{\omega}_i^2$ does not converge to ω^2 but in fact goes to zero suggesting that $\hat{\omega}$ should go to zero too. In fact, as Westerlund (2008) shows, if $q \to \infty$ with N, $T \to \infty$ and $\frac{q}{T} \to 0$,

$$\hat{\omega} \;=\; O_p\left(\frac{1}{\sqrt{q}}\right).$$

From Section 2 we know that $\tau_{LLC} = O_p(\sqrt{NT}\,)$ under H_{1a}, suggesting that τ_{LLC}^c is of the same order. Therefore, to determine the effect of the inconsistency of $\hat{\omega}^2$ on

$$\tau_{LLC}^c \;=\; \sqrt{2}\,\tau_{LLC} + \frac{NT}{\sqrt{2}}\frac{\hat{\omega}}{\sqrt{M_{22}^p}},$$

we only need to consider the second term, the bias, which under H_{1a} can be written as

$$\frac{NT}{\sqrt{2}}\frac{\hat{\omega}}{\sqrt{M_{22}^p}} \;=\; \frac{\sqrt{NT}}{\sqrt{2}}\,\hat{\omega}\,\frac{1}{\sqrt{\frac{1}{NT}M_{22}^p}} \;=\; \sqrt{NT}\,O_p\left(\frac{1}{\sqrt{q}}\right)O_p(1).$$

In order to appreciate the implications of this last result, suppose that q is set independent of T, so that the bias term is $O_p(\sqrt{NT}\,)$. The problem here is that while $\tau_{LLC} \to -\infty$, the bias term is diverging at the same rate but in the opposite direction, which means that τ_{LLC}^c is not a consistent test. The only way to make τ_{LLC}^c consistent is therefore to set q as a function of T, ensuring that the order of the bias term is lower than $O_p(\sqrt{NT}\,)$, and hence that $\tau_{LLC}^c \to -\infty$.

To illustrate these results Table 5 reports the size-adjusted power of τ_{LLC}^c for three different bandwidth selection rules. The first is the automatic rule of Newey and West (1994), while the other two are deterministic, and involve setting q either equal to $4(T/100)^{2/9}$ as suggested by Newey and West (1994) or equal to $3.21T^{1/3}$ as in LLC. The data are generated from (2), (4) and (5) with $c_i \sim U(0, 2c)$. In agreement with the results of Westerlund (2008) we see that the power can be very low and practically nonexisting unless $b = 3.21T^{1/3}$, which is also the most generous rule considered. For example, if $T = 100$, then $4(T/100)^{2/9} = 4$ while $3.21T^{1/3} \simeq 15$, an increase by almost a factor of four.

5 Cross section dependence

Fact 4: Cross-section dependence leads to deceptive inference

We consider two types of dependence, weak and strong. The first type refers to a situation in which all the eigenvalues of the covariance matrix of y_{it} are bounded as $N \to \infty$, which rules

Table 5: Size-adjusted power of the LLC test for different bandwidths.

			Bandwidth selection rule		
c	N	T	NW	$4(T/100)^{2/9}$	$3.21T^{1/3}$
5	10	100	3.9	3.9	6.2
	20	100	3.6	3.5	5.0
	10	200	1.9	1.6	3.5
	20	200	2.5	2.5	2.8
10	10	100	3.9	3.6	10.8
	20	100	3.7	3.7	7.9
	10	200	2.1	1.3	5.6
	20	200	2.7	1.8	3.9
20	10	100	7.2	6.0	38.5
	20	100	6.8	6.8	29.7
	10	200	5.2	1.6	20.6
	20	200	3.4	2.0	11.6
40	10	100	40.4	39.7	86.0
	20	100	27.3	28.3	85.2
	10	200	31.4	8.9	73.3
	20	200	18.5	4.2	65.2

Notes: The table reports the 5% size-adjusted rejection frequencies when the AR parameter is set to $\alpha_i = c_i/T\sqrt{N}$, where $c_i \sim U(0, 2c)$. NW refers to the automatic bandwidth selection rule of Newey and West (1994).

out the presence of unobserved common factors, but allows the cross-sectional units to be for example spatially correlated. The second type of dependence refers to a situation when at least one eigenvalue diverges with N, which arises when there are common factors present.

Suppose as in the previous section that (3) holds so that

$$\begin{pmatrix} \Delta y_{1t} \\ \vdots \\ \Delta y_{Nt} \end{pmatrix} = \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_N \end{pmatrix} \begin{pmatrix} y_{1t-1} \\ \vdots \\ y_{Nt-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \vdots \\ \varepsilon_{Nt} \end{pmatrix}$$

or, in matrix format,

$$\Delta y_t = \Pi y_{t-1} + \varepsilon_t.$$

However, instead of looking at the case when ε_{it} is dependent across t, we now consider the case when it is dependent across i.

In particular, let us begin by assuming that all eigenvalues of the covariance matrix

$$\Omega \;=\; \text{cov}(\varepsilon_t) \;=\; E(\varepsilon_t \varepsilon_t')$$

are bounded as $N \to \infty$, which means that the dependence is of the weak form. By the spectral decomposition, $\Omega = \Omega^{1/2}(\Omega^{1/2})' = V\Lambda V'$, where Λ is the diagonal matrix with the ordered eigenvalues $\lambda_1 \geq ... \geq \lambda_N$ along the diagonal, while V is the matrix of orthonormal eigenvectors. Let $y_t^* = \left(\begin{array}{ccc} y_{1t}^*, & \ldots, & y_{Nt}^* \end{array}\right)' = \Omega^{-1/2} y_t$, which under H_0 is nothing but a vector of uncorrelated random walks.

The above assumptions imply that M_{12} can be written as

$$M_{12} \;=\; \sum_{t=2}^{T} y_{t-1}' \Delta y_t \;=\; \sum_{t=2}^{T} (y_{t-1}^*)' \Lambda \Delta y_t^* \;=\; \sum_{i=1}^{N} \lambda_i \sum_{t=2}^{T} y_{it-1}^* \Delta y_{it}^* \;=\; \sum_{i=1}^{N} \lambda_i M_{12i}^*.$$

Let $\overline{\lambda^2} = \lim\limits_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \lambda_i^2$. By using the results of the previous sections,

$$\frac{1}{NT} M_{12} \;\xrightarrow{p}\; \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \lambda_i \frac{1}{T} E(M_{12i}^*) \;=\; \overline{\lambda} \frac{1}{T} E(M_{12i}^*) \;=\; 0$$

such that $\frac{1}{T\sqrt{N}} M_{12} \xrightarrow{d} \frac{1}{\sqrt{2}} \sqrt{\overline{\lambda^2}} \mathcal{N}(0,1)$ as $N, T \to \infty$. Similarly,

$$\frac{1}{NT^2} M_{22} \;=\; \frac{1}{NT^2} \sum_{t=2}^{T} y_{t-1}' y_{t-1} \;=\; \frac{1}{NT^2} \sum_{i=1}^{N} \lambda_i M_{22i}^* \;\xrightarrow{p}\; \overline{\lambda} \frac{1}{T^2} E(M_{22i}^*) \;\to\; \frac{1}{2} \overline{\lambda},$$

where the limit is taken as $N \to \infty$ followed by $T \to \infty$. This result, together with $\hat{\sigma}^2 \xrightarrow{p} \overline{\lambda}$, suggest that as $N, T \to \infty$

$$\tau_{LLC} \;=\; \frac{\frac{1}{T\sqrt{N}} M_{12}}{\hat{\sigma}\sqrt{\frac{1}{NT^2} M_{22}}} \;\xrightarrow{d}\; \frac{\frac{1}{\sqrt{2}} \sqrt{\overline{\lambda^2}} \mathcal{N}(0,1)}{\sqrt{\overline{\lambda}} \sqrt{\frac{\overline{\lambda}}{2}}} \;\sim\; \frac{\sqrt{\overline{\lambda^2}}}{\overline{\lambda}} \mathcal{N}(0,1),$$

which summarizes Theorem 1 of Breitung and Das (2005). In other words, if the dependence is weak then τ_{LLC} is still the asymptotically normal. However, as long as $\lambda_i \neq \lambda_j$ for at least some $i \neq j$, $\frac{\sqrt{\overline{\lambda^2}}}{\overline{\lambda}} > 1$ and so the variance will tend to increase with deceptive inference as a result. A similar result applies to τ_{IPS}. That is, the IPS test will also tend to be misleading in the presence of weak cross-section dependence.

These results suggest a simple correction that can be used to remove the effects of the weak dependence. Specifically, letting $v_1 \geq ... \geq v_N$ denote the eigenvalues of $\hat{\Omega}$, the estimated covariance matrix, then we have that as $N, T \to \infty$

$$\frac{\overline{v}}{\sqrt{\overline{v^2}}} \tau_{LLC} \;\xrightarrow{d}\; \mathcal{N}(0,1)$$

where $\overline{v^2} = \frac{1}{N}\sum_{i=1}^N v_i^2$ with an obvious definition of $\overline{v}$. Thus, by exploiting the asymptotic distribution of τ_{LLC} we can derive another test statistic whose asymptotic distribution is free of nuisance parameters and that has not been considered before.

In order to analyze the effects of strong dependence, suppose that

$$y_{it} \;=\; \theta_i f_t + y_{it}^s, \tag{11}$$

which can be written in matrix format as

$$y_t \;=\; \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_N \end{pmatrix} f_t + \begin{pmatrix} y_{1t}^s \\ \vdots \\ y_{Nt}^s \end{pmatrix} \;=\; \Theta f_t + y_t^s,$$

where y_{it}^s, the idiosyncratic component of y_{it}, again evolves according to (2) but now ε_{it} is independent across both i and t. The common factor f_t is assumed to be a scalar such that

$$f_t \;=\; \delta f_{t-1} + u_t, \tag{12}$$

where $|\,\delta\,| < 1$ and u_t is a mean zero and unit variance disturbance that is uncorrelated both across t and with ε_{it}. Under these conditions, and imposing H_0,

$$\Delta y_t \;=\; \Theta\Delta f_t + \Delta y_t^s \;=\; \Theta\big((\delta-1)f_t + u_t\big) + \varepsilon_t,$$

which in turn implies that

$$\Omega \;=\; E\big(\Delta y_t \Delta y_t'\big) \;=\; \frac{(1-\delta\,)^2}{1-\delta^2}\,\Theta\,\Theta' + \sigma^2 I_N.$$

The main difference here in comparison to the case with weak dependence is the presence of f_t, which suggests that the largest eigenvalue is no longer bounded but is in fact $O_p(N)$. Intuitively, the information regarding the common component Θf_t accumulates as we sum up the observations across i and therefore the largest eigenvalue will increase with N.

To see how the presence of f_t is going to change the previous results, write

$$\begin{aligned}\frac{1}{NT}M_{12} \;&=\; \frac{1}{NT}\sum_{t=2}^T(\Theta f_{t-1} + y_{t-1}^s)'(\Theta\Delta f_t + \varepsilon_t^s) \\ &=\; \frac{1}{NT}\sum_{t=2}^T\big(f_{t-1}\Theta'\Theta\Delta f_t + (y_{t-1}^s)'\varepsilon_t + f_{t-1}'\Theta'\varepsilon_t + (y_{t-1}^s)'\Theta\Delta f_t\big),\end{aligned}$$

where, letting $\mathcal{F}$ denote the sigma field generated by f_t,

$$\frac{1}{NT}\sum_{t=2}^T f_{t-1}'\Theta'\varepsilon_t \;=\; \frac{1}{NT}\sum_{i=1}^N\sum_{t=2}^T \theta_i f_{t-1}\varepsilon_{it} \;\xrightarrow{p}\; \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^N \theta_i \frac{1}{T}\sum_{t=2}^T f_{t-1}E(\varepsilon_{it}|\mathcal{F}) \;=\; 0$$

as $N \to \infty$, suggesting that $\frac{1}{\sqrt{NT}} \sum_{t=2}^{T} f'_{t-1}\Theta' \varepsilon_t = O_p(1)$. Moreover, since

$$\frac{1}{T\sqrt{N}} \sum_{t=2}^{T} (y^s_{t-1})' \varepsilon_t \;=\; O_p(1)$$

where $\frac{1}{T\sqrt{N}} \sum_{t=2}^{T} (y^s_{t-1})' \Theta \Delta f_t$ is of the same order, we obtain

$$\begin{aligned} \frac{1}{NT} M_{12} &= \frac{1}{NT} \sum_{t=2}^{T} f_{t-1}\Theta'\Theta \Delta f_t + O_p\left(\frac{1}{\sqrt{N}}\right) = \frac{1}{NT} \sum_{i=1}^{N}\sum_{t=2}^{T} \theta_i^2 f_{t-1}\Delta f_t + O_p\left(\frac{1}{\sqrt{N}}\right) \\ &\xrightarrow{p} -\frac{1-\delta}{1-\delta^2}\,\overline{\theta^2} \end{aligned}$$

as $N, T \to \infty$, where $\overline{\theta^2} = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \theta_i^2$.

Also,

$$\begin{aligned} \frac{1}{NT^2} M_{22} &= \frac{1}{NT^2} \sum_{t=2}^{T} (\Theta f_{t-1} + y^s_{t-1})'(\Theta f_{t-1} + y^s_{t-1}) \\ &= \frac{1}{NT^2} \sum_{t=2}^{T} (y^s_{t-1})' y^s_{t-1} + O_p\left(\frac{1}{T}\right) \xrightarrow{p} \frac{\sigma^2}{2}, \end{aligned}$$

where we have used that $\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} \theta_i^2 f^2_{t-1} \xrightarrow{p} \frac{(1-\delta)^2}{1-\delta^2}\,\overline{\theta^2}$ as $N, T \to \infty$, and that

$$\begin{aligned} \frac{1}{T\sqrt{N}} \sum_{t=2}^{T} (y^s_{t-1})' \Theta f_{t-1} &= \frac{1}{T\sqrt{N}} \sum_{i=1}^{N}\sum_{t=2}^{T} \theta_i y^s_{it-1} f_{t-1} = \frac{1}{T} \sum_{t=2}^{T} f_{t-1} \sum_{s=1}^{t} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \theta_i \varepsilon_{is} \\ &\xrightarrow{d} \frac{1}{T} \sum_{t=2}^{T} f_{t-1} \sum_{s=1}^{t} \sigma \sqrt{\overline{\theta^2}}\, \mathcal{N}(0,1) \end{aligned}$$

as $N \to \infty$, which is $O_p(1)$ as $T \to \infty$.

By collecting these results we obtain

$$\frac{\hat{\sigma}}{\sqrt{N}} \tau_{LLC} \;\xrightarrow{p}\; -\frac{\frac{1-\delta}{1-\delta^2}\,\overline{\theta^2}}{\sqrt{\frac{\sigma^2}{2}}},$$

or $\tau_{LLC} = O_p(\sqrt{N}\,)$ suggesting that the size of the LLC test will tend to one as $N \to \infty$.

As for the IPS test, note that

$$\begin{aligned} \frac{1}{T} M_{12i} &= \frac{1}{T} \sum_{t=2}^{T} y_{it-1}\Delta y_{it} = \frac{1}{T} \sum_{t=2}^{T} (\theta f_{t-1} + y^s_{it-1})(\theta_i \Delta f_t + \varepsilon_{it}) \\ &= \frac{1}{T} \sum_{t=2}^{T} y^s_{it-1}\varepsilon_{it} + \theta_i \frac{1}{T} \sum_{t=2}^{T} y^s_{it-1}\Delta f_t + \theta_i^2 \frac{1}{T} \sum_{t=2}^{T} f_{t-1}\Delta f_t + O_p\left(\frac{1}{\sqrt{T}}\right), \end{aligned}$$

while

$$\hat{\sigma}_i^2 \frac{1}{T^2} M_{22i} = \hat{\sigma}_i^2 \frac{1}{T^2} \sum_{t=2}^{T} y_{it-1}^2 = \hat{\sigma}_i^2 \frac{1}{T^2} \sum_{t=2}^{T} (\theta_i f_{t-1} + y_{it-1}^s)^2$$
$$= \hat{\sigma}_i^2 \frac{1}{T^2} \sum_{t=2}^{T} (y_{it-1}^s)^2 + O_p\left(\frac{1}{T}\right) = U_i + O_p\left(\frac{1}{T}\right).$$

Thus, by Taylor expansion,

$$\tau_i = \frac{M_{12i}}{\hat{\sigma}_i \sqrt{M_{22i}}} = \tau_i^s + \frac{\theta_i}{\sqrt{U_i}} \frac{1}{T} \sum_{t=2}^{T} y_{it-1}^s \Delta f_t + \frac{\theta_i^2}{\sqrt{U_i}} \frac{1}{T} \sum_{t=2}^{T} f_{t-1} \Delta f_t + O_p\left(\frac{1}{\sqrt{T}}\right),$$

where τ_i^s is the DF test based on y_{it}^s. Taking expectations, and then passing $T \to \infty$, we get

$$E(\tau_i|\mathcal{F}) = E(\tau_i^s|\mathcal{F}) + \frac{1}{T} \sum_{t=2}^{T} E\left(\frac{\theta_i}{\sqrt{U_i}} y_{it-1}^s | \mathcal{F}\right) \Delta f_t$$
$$+ E\left(\frac{\theta_i^2}{\sqrt{U_i}} | \mathcal{F}\right) \frac{1}{T} \sum_{t=2}^{T} f_{t-1} \Delta f_t + O_p\left(\frac{1}{\sqrt{T}}\right) \to \overline{C} \neq E(\tau),$$

where $\overline{C} < \infty$ henceforth denotes a generic real number. By subsequently passing $N \to \infty$,

$$\sqrt{\frac{\text{var}(\tau)}{N}} \tau_{IPS} = \frac{1}{N} \sum_{i=1}^{N} (\tau_i - E(\tau)) \xrightarrow{p} E\big(\tau_i - E(\tau)|\mathcal{F}\big) = \overline{C} - E(\tau) \neq 0,$$

showing that $\tau_{IPS} = O_p(\sqrt{N})$. Thus, just as with the LLC test the size of the IPS test will tend to one as $N \to \infty$.

Summarizing the results reported in this section we find that the presence of cross-section dependence is likely to lead to misleading inference. The extreme case being when the dependence is of the strong form, in which the test statistics actually become divergent. This last result is particularly interesting since in our setup f_t is stationary, and the presence of unit roots usually eliminates the effects of such variables. This is illustrated in Table 6, which depicts the size of the LLC and IPS tests in the presence of a single common factor. For simplicity the data are generated from (2) and (11) with no deterministic components or serial correlation. The factor is generated according to (10) with AR coefficient $\delta = 0$ and loading θ. The results show that the distortions are increasing in N, which is clearly in agreement with the theoretical predictions.

Myth 5: Sequential limits imply joint limits

As pointed out by Phillips and Moon (1999), the sequential limit theory, wherein T is passed to infinity before N, is very straightforward to apply and generally leads to quick asymptotic

Table 6: Size in the presence of a common factor.

θ	N	T	LLC	IPS
0	10	50	7.4	4.3
	20	50	6.1	3.3
	10	100	6.7	4.0
	20	100	6.1	4.6
1	10	50	29.7	41.4
	20	50	53.2	64.1
	10	100	47.3	59.9
	20	100	73.3	84.5
2	10	50	92.0	95.0
	20	50	99.4	99.8
	10	100	99.1	99.7
	20	100	100.0	100.0

Notes: The table reports the 5% rejection frequencies under H_0. θ refers to the factor loading.

results in a variety of settings. The main reason for this is that the passing of $T \to \infty$ while holding N fixed in the first step allows one to focus only on the first-order terms, as the higher order terms are eliminated prior to averaging over N. However, this feature can also be deceptive in its simplicity because it hides the need to control the relative expansion rate of the two dimensions. Indeed, as Phillips and Moon (1999) show, sequential convergence does not necessarily imply convergence in the joint limit as $N, T \to \infty$ simultaneously. In some situations the sequential limit theory may therefore break down. The problem is that the connection between the two methods is not well understood, and many researchers view breakdowns as extreme events.

Consider as an example the generalized least squares test of Breitung and Das (2005) and Harvey and Bates (2005),

$$\tau_{GLS} = \frac{\sum_{t=2}^{T} y_{t-1}' \hat{\Omega}^{-1} \Delta y_t}{\sqrt{\sum_{t=2}^{T} y_{t-1}' \hat{\Omega}^{-1} y_{t-1}}},$$

which is suitable for testing H_0 in the presence of weak cross-section dependence. Here y_t is again the vector stacking y_{it}, while $\hat{\Omega}$ is such that $\sqrt{T}\,(\hat{\Omega} - \Omega) = O_p(1)$.

We begin by deriving the sequential limit distribution of τ_{GLS}, and then we show that this distribution need not be the same as the one obtained when using joint limits.

By the consistency of $\hat{\Omega}$ and then Taylor expansion,

$$\hat{\Omega}^{-1} \;=\; \Omega^{-1} + O_p\left(\frac{1}{\sqrt{T}}\right).$$

Moreover, by a functional central limit theorem, $\frac{1}{\sqrt{T}} y_t \Rightarrow B(s)$ as $T \to \infty$, where $B(s) = \Omega^{1/2}W(s)$ and $W(s)$ is a vector standard Brownian motion. It follows that as $T \to \infty$ with N kept fixed,

$$\tau_{GLS} \;=\; \frac{\frac{1}{T}\sum_{t=2}^{T} y_{t-1}'\hat{\Omega}^{-1}\Delta y_t}{\sqrt{\frac{1}{T^2}\sum_{t=2}^{T} y_{t-1}'\hat{\Omega}^{-1}y_{t-1}}} \;\Rightarrow\; \frac{\int_0^1 B(s)'\Omega^{-1}dB(s)}{\sqrt{\int_0^1 B(s)'\Omega^{-1}B(s)ds}} \;=\; \frac{\int_0^1 W(s)'dW(s)}{\sqrt{\int_0^1 W(s)'W(s)ds}}.$$

But the elements of $W(s)$ are independent, suggesting that as $N \to \infty$

$$\frac{\int_0^1 W(s)'dW(s)}{\sqrt{\int_0^1 W(s)'W(s)ds}} \;=\; \frac{\frac{1}{\sqrt{N}}\int_0^1 W(s)'dW(s)}{\sqrt{\frac{1}{N}\int_0^1 W(s)'W(s)ds}} \;\xrightarrow{d}\; \mathcal{N}(0,1).$$

Thus, in the sequential limit as $T \to \infty$ and then $N \to \infty$

$$\tau_{GLS} \;\xrightarrow{d}\; \mathcal{N}(0,1).$$

Consider next the joint limit of the same test statistic. By using the results of Breitung and Das (2005),

$$\frac{1}{T\sqrt{N}}\sum_{t=2}^{T} y_{t-1}'\hat{\Omega}^{-1}\Delta y_t \;=\; \frac{1}{T\sqrt{N}}\sum_{t=2}^{T} y_{t-1}'\Omega^{-1}\Delta y_t + O_p\left(\frac{N}{\sqrt{T}}\right),$$

$$\frac{1}{NT^2}\sum_{t=2}^{T} y_{t-1}'\hat{\Omega}^{-1}y_{t-1} \;=\; \frac{1}{NT^2}\sum_{t=2}^{T} y_{t-1}'\Omega^{-1}y_{t-1} + O_p\left(\frac{\sqrt{N}}{\sqrt{T}}\right)$$

as N, $T \to \infty$, from which it follows that

$$\begin{aligned}\tau_{GLS} \;&=\; \frac{\frac{1}{T\sqrt{N}}\sum_{t=2}^{T} y_{t-1}'\hat{\Omega}^{-1}\Delta y_t}{\sqrt{\frac{1}{NT^2}\sum_{t=2}^{T} y_{t-1}'\hat{\Omega}^{-1}y_{t-1}}} \;=\; \frac{\frac{1}{T\sqrt{N}}\sum_{t=2}^{T} y_{t-1}'\Omega^{-1}\Delta y_t}{\sqrt{\frac{1}{NT^2}\sum_{t=2}^{T} y_{t-1}'\Omega^{-1}y_{t-1}}} + O_p\left(\frac{N}{\sqrt{T}}\right)\\ &=\; \frac{\frac{1}{T\sqrt{N}}\sum_{t=2}^{T}(y_{t-1}^*)'\Delta y_t^*}{\sqrt{\frac{1}{NT^2}\sum_{t=2}^{T}(y_{t-1}^*)'y_{t-1}^*}} + O_p\left(\frac{N}{\sqrt{T}}\right),\end{aligned}$$

where y_t^* is again a vector of uncorrelated random walks. As for the first term on the right-hand side, it is easily seen that

$$\frac{\frac{1}{T\sqrt{N}}\sum_{t=2}^{T}(y_{t-1}^*)'\Delta y_t^*}{\sqrt{\frac{1}{NT^2}\sum_{t=2}^{T}(y_{t-1}^*)'y_{t-1}^*}} \;\xrightarrow{d}\; \frac{\frac{1}{\sqrt{2}}\mathcal{N}(0,1)}{\sqrt{\frac{1}{2}}} \;\sim\; \mathcal{N}(0,1).$$

Thus, only if we assume that $\frac{N}{\sqrt{T}} \to 0$ as $N, T \to \infty$ do we end up with the same asymptotic distribution as before. In other words, although this does not turn up in the derivations, the sequential limit is not going to work if $N^2 >> T$. This makes sense even from an empirical point of view, as $\hat{\Omega}$ is singular unless $T \geq N$, a fact not accounted for when using the sequential limit method. It also explains the poor small sample properties of the test if T is small relative to N^2, as documented by Breitung and Das (2005).

Of course, this example is still quite specific, which makes it difficult to appreciate the generality of the critique. Let us therefore consider another example. In particular, let us reconsider the limiting null distribution of τ_{IPS} under the assumption that ε_{it} is normal, independent and identically distributed, in which case we know from Phillips (1987) that

$$\frac{1}{\sqrt{T}} y_{it} = \sigma W_i(s) + O_p\left(\frac{1}{T}\right).$$

Since $\sqrt{T}(\hat{\sigma}_i^2 - \sigma^2) = O_p(1)$ from LLC, we obtain

$$\tau_i = \frac{M_{12i}}{\hat{\sigma}_i \sqrt{M_{22i}}} = \frac{M_{12i}^{\circ}}{\sqrt{M_{22i}^{\circ}}} + O_p\left(\frac{1}{T}\right),$$

implying that

$$\begin{aligned}\tau_{IPS} &= \frac{\sqrt{N}(\overline{\tau} - E(\tau))}{\sqrt{\text{var}(\tau)}} = \frac{1}{\sqrt{N\,\text{var}(\tau)}} \sum_{i=1}^{N} (\tau_i - E(\tau)) \\ &= \frac{1}{\sqrt{N\,\text{var}(\tau)}} \sum_{i=1}^{N} \left(\frac{M_{12i}^{\circ}}{\sqrt{M_{22i}^{\circ}}} - E(\tau) \right) + O_p\left(\frac{\sqrt{N}}{T}\right),\end{aligned}$$

where the first term on the right-hand side converges to a standard normal distribution as $N \to \infty$. Thus, for τ_{IPS} to be asymptotically normal, one needs to assume that $\frac{\sqrt{N}}{T} \to 0$ as $N, T \to \infty$, as otherwise the $O_p(\sqrt{N}/T)$ remainder will not disappear. A similar result applies to τ_{LLC}. The point here is that if we use the sequential limit method where N is treated as fixed in the first step then this remainder is $O_p(1/T)$, which vanishes as $T \to \infty$. The sequential limit method therefore breaks down unless $\frac{\sqrt{N}}{T} \to 0$. But if this result holds in the current very restrictive case with normal innovations, it is expected to hold also under more relaxed conditions. The risk of breakdown of the sequential limit method is therefore more of a rule rather than an exception.

Figures 3 and 4 illustrate this point by plotting the size of the LLC and IPS tests at the 5% level when T is set as a function of N. If $T = N$ the condition that $\frac{\sqrt{N}}{T} \to 0$ is satisfied, while if $T = \sqrt{N}$, then the condition is violated. The model includes a constant but otherwise

Figure 4: Size of τ_{LLC} when T is set as a function of N.

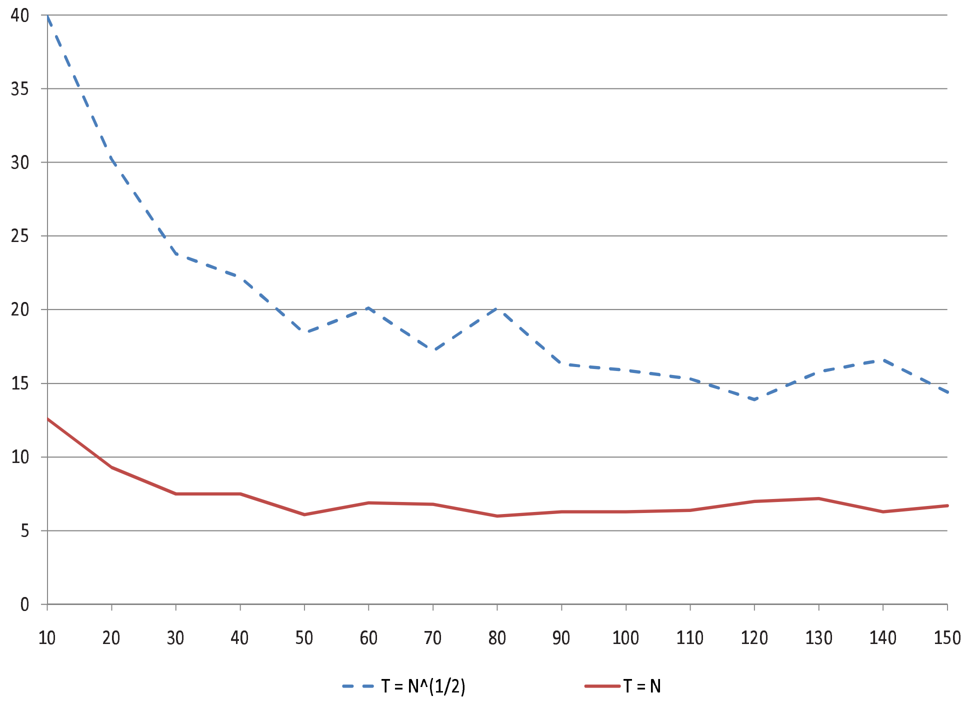
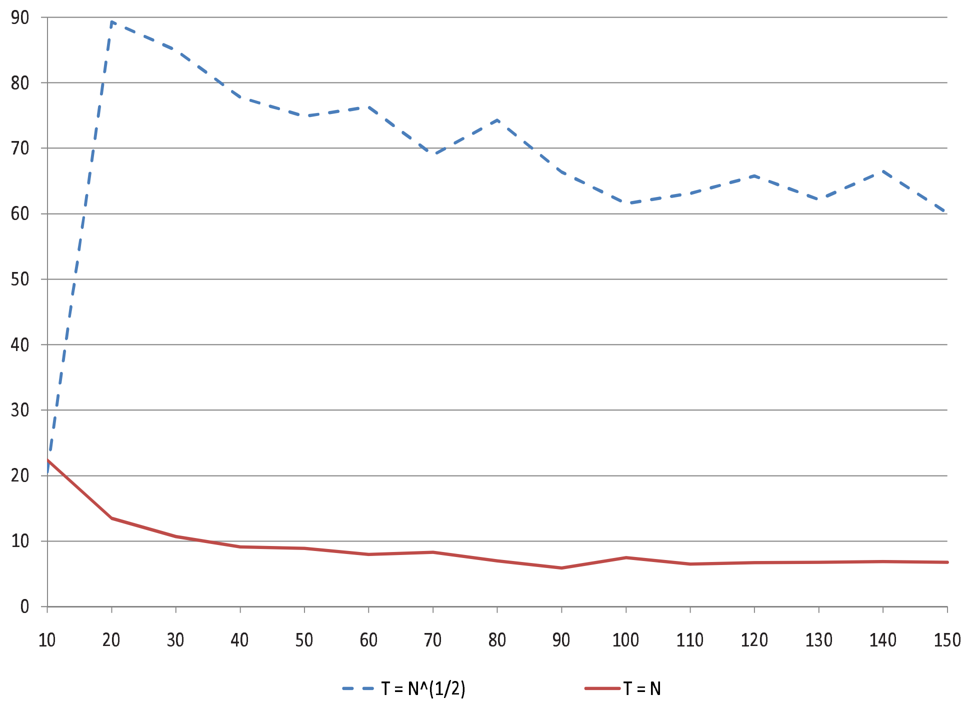


Figure 5: Size of τ_{IPS} when T is set as a function of N.



there are no nuisance parameters to correct for. In contrast to the case when $T = N$, in which both tests tend to perform well, we see that setting $T = \sqrt{N}$ generally leads to substantial size distortions, especially for the IPS test.

Fact 5: The IPS and LLC tests fail under cross-unit cointegration

Suppose that (10) holds, and that $\alpha_i < 0$ for all i, while $\delta = 1$ so that f_t is the only source of non-stationarity in y_{it}. Under these conditions, y_{it} and y_{jt} are cointegrated, a situation commonly referred to as cross-unit cointegration.

By analogy to the case when $|\,\delta\,| < 1$,

$$
\begin{aligned}
\frac{1}{NT} M_{12} &= \frac{1}{NT} \sum_{t=2}^{T} (\Theta f_{t-1} + y_{t-1}^s)'(\Theta u_t + \varepsilon_t^s) \\
&= \frac{1}{NT} \sum_{t=2}^{T} \left(f_{t-1}\Theta'\Theta u_t + (y_{t-1}^s)'\varepsilon_t + f_{t-1}'\Theta'\varepsilon_t + (y_{t-1}^s)'\Theta u_t \right) \\
&= \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} \theta_i^2 f_{t-1} u_t + O_p\left(\frac{1}{\sqrt{N}}\right) \;\Rightarrow\; \overline{\theta^2} \int_0^1 W(s) dW(s)
\end{aligned}
$$

as $N, T \to \infty$, where $W(s)$ is now a scalar standard Brownian motion. Moreover,

$$
\begin{aligned}
\frac{1}{NT^2} M_{22} &= \frac{1}{NT^2} \sum_{t=2}^{T} (\Theta f_{t-1} + y_{t-1}^s)'(\Theta f_{t-1} + y_{t-1}^s) \\
&= \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} \theta_i^2 f_{t-1}^2 + O_p\left(\frac{1}{\sqrt{N}}\right) \;\Rightarrow\; \overline{\theta^2} \int_0^1 W(s)^2 ds,
\end{aligned}
$$

showing that

$$
\frac{\hat{\sigma}}{\sqrt{N}} \tau_{LLC} \;\Rightarrow\; \frac{\overline{\theta^2} \int_0^1 W(s) dW(s)}{\sqrt{\overline{\theta^2} \int_0^1 W(s)^2 ds}}
$$

or $\tau_{LLC} = O_p(\sqrt{N}\,)$.

Remember that if there is no cross-dependence, $\tau_{LLC} \xrightarrow{d} \mathcal{N}(0,1)$. Thus, for this test the left-tail critical value at the 5% level is given by -1.645. But the size is given by

$$
\begin{aligned}
\lim_{N,T\to\infty} P(\tau_{LLC} < -1.645) &= \lim_{N,T\to\infty} P\left(\frac{1}{\sqrt{N}} \tau_{LLC} < -\frac{1.645}{\sqrt{N}}\right) \\
&= P\left(\int_0^1 W(s) dW(s) < 0\right) \;\simeq\; 0.7,
\end{aligned}
$$

see Breitung and Das (2008). In other words, the size of τ_{LLC} is upwards biased in the presence of cross-unit cointegration.

The analysis of τ_{IPS} is very similar to the case when $|\,\delta\,| < 1$. We begin by noting that

$$\begin{aligned}
\frac{1}{T}\,M_{12i} &= \frac{1}{T}\sum_{t=2}^{T}(\theta_i f_{t-1} + y^s_{it-1})(\theta_i u_t + \Delta y^s_{it}) \\
&= \theta_i^2\,\frac{1}{T}\sum_{t=2}^{T} f_{t-1}u_t + \theta_i\,\frac{1}{T}\sum_{t=2}^{T} f_{t-1}\Delta y^s_{it} + \frac{1}{T}\sum_{t=2}^{T} y^s_{t-1}\Delta y^s_{it} + O_p\left(\frac{1}{\sqrt{T}}\right), \\
\hat{\sigma}_i^2\,\frac{1}{T^2}\,M_{22i} &= \hat{\sigma}_i^2\,\frac{1}{T^2}\sum_{t=2}^{T}(\theta_i f_{t-1} + y^s_{it-1})^2 \;=\; \hat{\sigma}_i^2\,\frac{1}{T^2}\sum_{t=2}^{T}\theta_i^2 f_{t-1}^2 + O_p\left(\frac{1}{T}\right) \\
&= V_i + O_p\left(\frac{1}{T}\right).
\end{aligned}$$

Thus, using τ^f to denote the DF test based on f_t,

$$\tau_i \;=\; \tau^f + \frac{\theta_i}{\sqrt{V_i}}\,\frac{1}{T}\sum_{t=2}^{T} f_{t-1}\Delta y^s_{it} + \frac{1}{\sqrt{V_i}}\,\frac{1}{T}\sum_{t=2}^{T} y^s_{t-1}\Delta y^s_{it} + O_p\left(\frac{1}{\sqrt{T}}\right).$$

Clearly, $E(\tau_i|\mathcal{F}) \to \overline{C} \neq E(\tau)$, where $\overline{C}$ again denotes a real positive number, and so

$$\sqrt{\frac{\mathrm{var}(\tau)}{N}}\,\tau_{IPS} \;=\; \frac{1}{N}\sum_{i=1}^{N}(\tau_i - E(\tau)) \;\xrightarrow{p}\; \overline{C} - E(\tau) \;\neq\; 0,$$

which shows that $\tau_{IPS} = O_p(\sqrt{N}\,)$.

Consequently, the presence of cross-unit cointegrating relationships causes the IPS and LLC statistics to become divergent, which is in agreement with the simulation results of Banerjee *at al.* (2005), showing that the presence of such relationships can lead to severe size distortions.

Myth 6: Factor based tests are based on very relaxed assumptions

An important feature of the factor model in (10) is that f_t and y^s_{it} can have different orders of integration, see for example Bai and Ng (2008). In most other work on panel unit root tests with common factors this is not the case. In particular, consider the data generating process adopted by Moon and Perron (2004), Moon *et al.* (2007), Pesaran (2007) and Phillips and Sul (2003), which in the case with a single factor and no deterministic components can be written as

$$y_{it} \;=\; \rho_i y_{it-1} + w_{it}$$

with $w_{it} = \theta_i g_t + v_{it}$, where g_t and v_{it} are independent of each other as well as across both i and t. This specification differs from (10) in that it essentially specifies the dynamics of the observed series, whereas (10) specifies the dynamics of unobserved components. Assuming $y_{i0} = 0$ and $\rho_i = \rho$ for all i, the above model can be written in terms of (10) as follows

$$y_{it} \;=\; \rho_i y_{it-1} + \theta_i g_t + v_{it} \;=\; \theta_i(\rho f_{t-1} + g_t) + (\rho y^s_{it-1} + v_{it}) \;=\; \theta_i f_t + y^s_{it}.$$

It follows that if $\rho = 1$, then $f_t = f_{t-1} + g_t$ and $y^s_{it} = y^s_{it-1} + v_{it}$, and both variables are non-stationary. Conversely, if $|\,\rho\,| < 1$, then both variables are stationary. The common and idiosyncratic components of the above model are therefore restricted to have the same order of integration. Note that when ρ_i is heterogeneous, then the above model cannot be expressed in terms of (10). But under the null that $\rho_i = 1$ for all i, then it is nested in (10). One study that explicitly takes (10) as the data generating process is that of Bai and Ng (2004). This study is therefore less restrictive than the above mentioned studies.

An even more general approach is proposed by Breitung and Das (2008), who treat the above AR model as a reduced form regression where w_{it} does not necessarily has to have a common factor structure. One important advantage of this approach is that no factor structure needs to be estimated.

6 Concluding remarks

This paper points to six myths and five facts that are oftentimes overlooked when considering the problem of testing for a unit root in panel data. Suppose for example that one would like to test the null hypothesis that the variable y_{it} has a unit root versus the alternative that it is stationary with a nonzero mean, a very common research scenario. The by far most common way of carrying out such a test is to use the traditional DF approach of applying least squares to an intercept-augmented autoregression. Being so common it is easy to get the impression that demeaning in this way is the best way to accommodate the nonzero mean in y_{it}. But this is only a myth. Indeed, as we show in the paper the inclusion of the intercept introduces a bias in the estimated AR coefficient, which then has to be corrected somehow. However, in so doing we find that the resulting corrected test is likely to suffer from low power and may even become inconsistent in some circumstances. As a response to this a few alternative demeaning procedures are suggested.

In this example, although applying traditional time series techniques leads to a larger computational burden and poorer small-sample performance, usually there are no fundamental shortcomings or flawed inference. Unfortunately, this is not always the case. Quite on the contrary, we find that ignoring these myths and facts will in many cases lead to serious side-effects, including a complete breakdown of the whole test procedure. One example of such a situation is when y_{it} is contaminated by cross-section dependence in the form of common factors, in which case a failure to account for these factors can cause the test statistic to become divergent.

The implication is that one should be careful not to approach the testing problem from a too narrow and stylized perspective. In particular, we believe that the usual practice of looking at the problem from mainly a time series perspective can be deceptive in its simplicity, typically ignoring many important issues such as cross-sectional dependence, incidental trends and joint limit restrictions.

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