Equilibrium Existence in the Linear-city Model of Spatial Competition Revisited♣

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Abstract:

In this paper we studied the existence of the sequential first-location-then-price equilibrium in the linear-city model of spatial competition. Using a generic transport cost function which generalizes the convex and the concave case, we find that the demand function is always connected in the convex case whereas in the concave case this function can be non-connected. A crucial change of variable that permit the profit functions of both firms being symmetric allows us to give a general result which characterize the exact regions of location pairs for which a price equilibrium exist. This solution closes the analysis on the existence of equilibrium for the class of linear-quadratic and convex transport cost function.

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1. Introducción

The main purpose of this paper is to clarify and extend some important issues concerning existence and properties of equilibria in the horizontal differentiation model "*a la* Hotelling".

In the classical Hotelling (1929)'s model of product differentiation, two firms and a continuum of consumers locate along a main street, consumers make choices based on the price of the product plus the transport cost, which depends on the distance to the firm. In this setting, the typical equilibrium concept is the sequential (or sub-game perfect) equilibrium, where firms first simultaneously locate, and then simultaneously set the price of the product which maximize individual profits.

A deeply explored research issue in this framework has been to analyze the existence and uniqueness of the (pure strategy) sequential equilibrium under alternative specifications of the transport cost function. The motivation for this was Hotelling (1929)'s initial claim that an equilibrium involving minimum differentiation existed under linear transport costs, and D'Aspremont *et al.* (1979) response showing the non-existence of the equilibrium in that case, and the existence of a unique equilibrium involving maximum differentiation, if transport costs were quadratic in distance.

The linear-quadratic class of transport cost functions includes the two specifications mentioned above. There, transport costs are represented by the function $c(d) = ad + bd^2$, where d is the distance between the consumer and the firm. Under this cost structure, it is well known that there exists no (pure strategy) price equilibrium for all the possible locations of the firms. For example, Gabszewicz and Thisse (1986) analyze the case where transport costs are convex in distance (that is, the case where $b > 0$), and show that there does not exist a price equilibrium for all the possible locations of the firms. In fact, a price equilibrium can exist only if firms are sufficiently far one from each other. A crucial assumption of their analysis is the fact that firms locate symmetrically with respect to the centre of the city. Anderson (1988) extends the analysis to the case of asymmetric locations, and identifies the pairs of locations that do not satisfy the necessary requirements for the price equilibrium existence. As long as $a > 0$, he finds that for any location of one of the firms, there exists some location of the other firm for which there is no price equilibrium. Interestingly, Anderson (1988) finds that these equilibrium necessary conditions can be satisfied by sufficiently close locations in one extreme of the line, a result which clearly contrasts with that of Gabszewicz and Thisse (1986). However, Anderson (1988) does not check equilibrium sufficient conditions; in fact, his goal is to prove non-existence of a (pure strategies) equilibrium (and afterwards analyze the case of mixed strategies), but not to characterize the equilibrium regions (see his Proposition 1, p. 485).

Recently, Hamoudi and Moral (2005) have incorporated the concave specification into the debate ($b < 0$), and have shown that the sequential equilibrium does not exist in that case either, particularly when $a = -b$. Moreover, they have computed the equilibrium regions comparing both the concave and the convex cases, and have found that the equilibrium region in the concave case is even lower than that of the convex case. These equilibrium regions consist of pairs of locations sufficiently far one from each other. From Hamoudi and Moral (2005), we might infer that close enough location pairs (the ones shown in Anderson, 1988) do not satisfy the sufficient conditions for the existence of the equilibrium.

More recently, Arguedas and Hamoudi (2008) have studied the existence of the sequential first-location-then-price equilibrium in the linear model of product differentiation when transport costs are concave linear-quadratic in distance. They analyze the equilibrium in the vertical and horizontal differentiation cases. In the former case, they show the existence and uniqueness of perfect equilibrium, whereas in the other case they find the necessary and sufficient conditions for a price equilibrium to exist.

We propose in this paper a variant of the traditional Hotelling model of spatial competition. In our formulation, the transport cost structure is the key feature of the model. We assume a transport cost function which generalizes at the same time the convex and the concave case. However, in order to study perfect Nash price equilibrium, we specify a particular transport cost function, the class of linear-quadratic and convex transport cost function.

Our results confirm the general property that the sequential equilibrium fails to exist under linear-quadratic transport costs, in line with Gabszewicz and Thisse (1986), Anderson (1988), Hamoudi and Moral (2005) and Arguedas and Hamoudi (2008). The reason is that no price equilibrium exists for all the possible locations of the firms. As these authors, we confirm the existence of price equilibrium when firms locate sufficiently far in the case where the indifference consumer is located between the firms. Furthermore, we generalize the particular analysis of Anderson (1988), which only considers that one firm is located in the extreme of the city, by studying the equilibrium existence at any firm's location. A price equilibrium exists if firms are located sufficiently close one from each other. To the point, our main contribution is to characterize the exact regions of location pairs for which a price equilibrium exist and, in this sense, we generalize the previous results of these authors.

The literature on product differentiation is vast, see Brenner (2001) for an overview. Several variations of the model not included here refer to the consideration of mixed strategies equilibria (but see Osborne and Pitchik, 1987 or Anderson, 1988), alternative transport cost specifications (Economides, 1986), the circle model (Anderson, 1986 or De Frutos *et al.*, 1999, 2002), alternative consumers densities (Anderson and Goeree, 1997) or heterogeneous consumers' transport costs (Egli, 2007), among others.

The remainder of the paper is organized as follows. Section 2 describes the model and Section 3 presents the demands that firms attract under a general tractor cost function. In

Section 4, we obtain the corresponding price equilibrium regions. Section 5 shows the main conclusions.

2. The model

We consider the well known Hotelling's location model but we remove the assumption that the transport cost function is linear in distance. The basic scenario is as follows. There are two firms, labelled *X* and *Y*, selling an homogeneous product. As it is usual in the literature, we assume zero production costs. Firms are located at $x, y \in [0,1]$ and $x \leq y$ and they charge mill-prices p_x and p_y, given their locations.

Consumers are uniformly distributed along the market. Each buys just one unit of the industry good at the firm with lower full prices, made up of the product price plus the transport cost. Let $\alpha \in [0,1]$ denotes the consumer location in the linear market. The distance between the consumer and the seller is defined by $d_s = |\alpha - s|$ where $s = x, y$.

we consider a general transport costs function $c(d_s)$, such that $c'(d_s) > 0$ and $c''(d_s) \geq 0$ or $c''(d_s) \leq 0$ i.e. the function can be convex or concave. In order to study the equilibrium we consider the convex linear quadratic transport cost function introduced by Gabszewicz and Thisse (1986):[1]

$$c(d_s) = ad_s + bd_s^2, \quad \forall\ s = x, y \tag{1}$$

where *a* and *b* are non-negative parameters[2].

In this model the solution concept is the sequential equilibrium. In the first stage firms *X* and *Y* simultaneously chose their locations at *x* and *y*, an then simultaneously set the price of the product.

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[1] The concave case (b < 0) has been studied by Arguedas and Hamoudi (2008).

[2] When *a* > 0 and *b* = 0 we have the Hotelling's classic model with linear transportation costs.

3. Transport cost and demand structure

We can determine the demand addressed to each firm by looking for the indifferent consumer i. e. the consumer who faces the same full price of both companies. For this consumer:

$$p_x + c(d_x) = p_y + c(d_y) \qquad (2)$$

We can find an indifferent consumer in regions $[0, x]$, $[x, y]$ or $[y,1]$ depending on the prices and the firms locations. Once we identify this consumer, we immediately know that consumers located to his left are served by one firm, while consumers located to his right are served by the other. The condition to find the indifferent consumer in those three regions is established in the following result:

***Lemma* 1.** *One indifferent consumer is located in:*

(i) $[0, x]$ *if and only if* $p_x - p_y \in [Min\{c(y-x), c(y) - c(x)\}, Max\{c(y-x), c(y) - c(x)\}]$

(ii) $[x, y]$ *if and only if* $p_x - p_y \in [-c(y-x),\ c(y-x)]$

(iii) $[y,1]$ *if and only if*

$p_x - p_y \in [Min\{-c(y-x), c(1-y) - c(1-x)\}, Max\{-c(y-x), c(1-y) - c(1-x)\}]$

Proof:

(i) If the indifferent consumer locates at $[0, x]$ then the consumer located at 0 strictly prefer one firm, while the consumer located at *x* strictly prefers the other. If the consumer located at 0 prefers firm *X*, it is because $p_x - p_y < c(y) - c(x)$. But then, the consumer located at *x* must prefer firm *Y*, and therefore, $p_x - p_y < c(y-x)$. Consequently, it must be the case that $c(y-x) < p_x - p_y < c(y) - c(x)$, which is feasible if and only if transport costs are strictly convex, since $c(y-x) < c(y) - c(x)$. Conversely, if transport costs are strictly concave in distance, we then have $c(y) - c(x) < c(y-x)$. There can exist an indifferent consumer in this region if and only if $c(y)-c(x)< p_x - p_y < c(y-x)$, which then means that the consumer located at 0 prefers firm *Y*, while the consumer located at *x* prefers firm *X*. Summing up both possibilities, we then have that an indifferent consumer exists in region $[0,x]$ if and only if $p_x - p_y \in [Min\{c(y-x),c(y)-c(x)\}, Max\{c(y-x),c(y)-c(x)\}]$.

(ii) If the indifferent consumer is located at $[x,y]$, then the consumer located at *x* strictly prefers one firm, while the consumer located at *y* prefers the other. The consumer located at *x* prefers firm *X* if and only if $p_x - p_y < c(y-x)$, while the consumer located at *y* prefers firm *Y* if and only if $-c(y-x) < p_x - p_y$. Since $-c(y-x) < c(y-x)$, we consequently have $-c(y-x) < p_x - p_y < c(y-x)$, independently of the shape of transport costs.

(iii) Using an analogous procedure to that of part (i), we can easily conclude that the consumer located at *y* prefers firm *X* (*Y*) and the consumer located at 1 prefers firm *Y* (*X*) under convex (concave) transport costs, and the condition for the price difference is the desired one. ■

For future references we denote α_1^L the location of the indifferent consumer in region $[0,x]$, α^C the location of the indifferent consumer in region $[x,y]$, and α_2^L the location of the indifferent consumer in region $[y,1]$.

The intuition of this result is simple. Assume, for instance, that an indifference consumer is located at α_1^L. This means that the consumer located at 0 prefers one of the firms, while the consumer located at *x* prefers the other firm. If transport costs are strictly convex in distance, the consumer located at 0 prefers firm *X*, since $p_x - p_y < c(y)-c(x)$, while the consumer located at *x* prefers firm *Y*, since. Conversely, if transport costs are strictly concave in distance, the consumer that lives at 0 prefers firm *Y*, while the consumer that lives at *x* prefers firm *X*.[3]

If the indifferent consumer is located at α^C, it is because the consumer located at *x* prefers one of the firms, while the consumer located at *y* prefers the other firm. In this case, independently of the transport costs being either convex or concave, the consumer located at *x* prefers firm *X*, while the consumer located at *y* prefers firm *Y*, and the difference in prices $p_x - p_y$ must lie in the interval $[-c(y-x), c(y-x)]$.

Finally, if the indifference consumer is located at α_2^L, the consumer located at *y* prefers firm *X* (*Y*) under strictly convex (concave) transport costs, while the consumer located at *x* prefers firm *Y* (*X*).

However, depending on the shape of the transport cost function, we can find one or two indifferent consumers as we show in the following result:

***Lemma* 2.** *(i) Under strictly convex transport costs, there exists a unique indifferent consumer (that can be* α_1^L *or* α^C *or* α_2^L*) if and only if* $p_x - p_y \in [c(1-y)-c(1-x),\ c(y)-c(x)]$. *Else, only one firm attracts all the demand.*

(ii) Under strictly concave transport costs, there are two indifferent consumers (that can be α_1^L and α^C*, or* α^C *and* α_2^L*) if and only if either* $p_x - p_y \in [-c(y-x), c(1-y)-c(1-x)]$ *or* $p_x - p_y \in [c(y)-c(x), c(y-x)]$ *and a unique indifferent consumer (*α^C*) if and only if* $p_x - p_y \in [c(1-y)-c(1-x), c(y)-c(x)]$. *Else, only one firm attracts all the demand.*

Proof:

Under strictly convex transport costs, the following relationship holds:

$$c(1-y)-c(1-x) < -c(y-x) < c(y-x) < c(y)-c(x).$$

[3] Note that $c(y-x) < (>) c(y)-c(x)$ if and only if transport costs are strictly convex (concave).

Therefore, combining this relationship with Lemma 1, there exists one and only one indifferent consumer if and only if $p_x - p_y \in [c(1-y)-c(1-x), c(y)-c(x)]$. Else, only one firm serves all the demand.

Under strictly concave transport costs, the following relationship holds:

$$-c(y-x) < c(1-y)-c(1-x) < c(y)-c(x) < c(y-x).$$

Combining this relationship with the result of Lemma 1, note that there exist two indifferent consumers when either $p_x - p_y \in [-c(y-x), c(1-y)-c(1-x)]$ (one at $[x,y]$ and another one at $[y,1]$) or when $p_x - p_y \in [c(y)-c(x), c(y-x)]$ (one at $[x,y]$ and another one at $[0,x]$), and only one indifferent consumer when $p_x - p_y \in [c(1-y)-c(1-x), c(y)-c(x)]$ located at $[x,y]$. In the remaining possibilities, only one firm serves all the demand. ■

In the convex case, there can not exist more than an indifferent consumer (located in either $[0,x]$, $[x,y]$ or $[y,1]$, depending on the price differences and the locations of the firms). In this case, all the consumers located between 0 and the position of the indifferent consumer (i.e., those located to the left of the indifferent consumer), prefer firm *X*, while the remaining consumers prefer firm *Y*.

Interestingly, under concave transport costs, if there exists only one indifferent consumer, she must be necessarily located in the central region $[x,y]$. In this case, consumers to the left of α^C prefer firm *X*, while consumers to the right prefer firm *Y* (at least, those consumers located sufficiently close to α^C). But moreover, there can be a second indifferent consumer in the lateral regions, either the lateral 1, $[0,x]$ or the lateral 2, $[y,1]$. If, for example, that second indifferent consumer were located in $[0,x]$ (i.e., in position α_1^L), this means that consumers located to the left of α_1^L prefer firm *Y*. Therefore, the demand of firm *Y* is non-connected, because firm *Y* attracts the consumers located to the right of α^C and those located to the left of α_1^L, where $\alpha_1^L < \alpha^C$. In other words, firm *Y* attracts close consumers and very far consumers, but not consumers located at an intermediate distance (i.e., those located between α_1^L and α^C). This can occur only if the price reduction offered by firm *Y* (as compared to that of firm *X*) is attractive enough to compensate the additional transport costs of the furthest consumers (travelling from firm *X* to firm *Y*); or, put differently, if the additional transport costs of the furthest consumers is small enough, that is, when transport costs are sufficiently concave.

To understand the precise concept of demand connectedness, consider the following example. Suppose that the two firms *X* and *Y*, and three arbitrary consumers 1, 2 and 3 are located along a main street, as depicted in Figure 1.

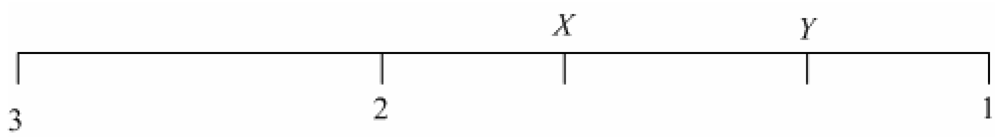

Figure 1. Example of firms and consumer locations

Suppose that consumers 1 and 2 live within walking distance from firms *X* and *Y*, respectively, whereas consumer 3 lives far from both firms, but closer to firm *X*. Clearly, firm *Y* can attract consumers 1 and 3 only if the reduction of the price (compared to that of firm *X*) outweighs the additional transport cost from firm *X* to firm *Y*. Probably, consumer 1 would need a car only if she chooses firm *Y* while consumer 3 would need it anyway. Consumer 1's additional transportation cost of travelling from firm *X* to firm *Y* is then larger than that of consumer 3. Thus, the price reduction in firm *Y* may be attractive only to consumers 2 and 3, but not to consumer 1. In this case, the demand of firm *Y* is non-connected since consumer 1 (who lives somewhere between consumers 2 and 3) does prefer firm *X*.

This example illustrates a situation that can occur only under concave transport costs: consumer 3's additional transport cost of travelling from firm *X* to firm *Y* is smaller than that of consumer 1. Under convex transport costs, however, demands are connected always, since in that case, consumer 3's additional cost of travelling from *X* to *Y* is larger than that of consumer 1. Therefore, *Y* could attract either consumer 2 only, or consumers 1 and 2, or the three consumers, but not consumers 2 and 3 only.

Given the existence conditions of the indifference consumers we can simultaneously derive the demand functions for convex and concave transport cost:

$$\text{Demand} = \begin{cases} \text{Convex case} & \text{Concave case} & & \\ 1 & 1 & \text{if} & p_x - p_y \in R^{-\infty} \\ \alpha_2^L & \alpha^C + (1-\alpha_2^L) & \text{if} & p_x - p_y \in R_2^L \\ \alpha^C & \alpha^C & \text{if} & p_x - p_y \in R^C \\ \alpha_1^L & \alpha^C - \alpha_1^L & \text{if} & p_x - p_y \in R_1^L \\ 0 & 0 & \text{if} & p_x - p_y \in R^{+\infty} \end{cases} \qquad (3)$$

where

$$R^{-\infty} = \left[-\infty, Min\{-c(y-x), c(1-y)-c(1-x)\}\right]$$

$$R_2^L = \left[Min\{-c(y-x), c(1-y)-c(1-x)\}, Max\{c(y-x), c(y)-c(x)\}\right]$$

$$R^C = \left[Max\{c(y-x), c(y)-c(x)\}, Min\{-c(y-x), c(1-y)-c(1-x)\},\right]$$

$$R_1^L = \left[Min\{-c(y-x), c(1-y)-c(1-x)\},, Max\{-c(y-x), c(1-y)-c(1-x)\}\right]$$

$$R^{+\infty} = \left[Max\{-c(y-x), c(1-y)-c(1-x)\}, +\infty\right]$$

As we can observe, independently of the particular expression of the transport cost function, the demand function is always connected in the convex case whereas in the concave case this function can be non-connected.

4. Price equilibrium existence

In this section we analyze the price equilibrium existence assuming that the transport cost function is the linear-quadratic convex function introduced by Gabszewicz and Thisse (1986), $c(d_s) = ad_s + bd_s^2$ with $s = x, y$ and $a, b > 0$.[4] We also evaluate the general demand function (3) for this transport cost function, to obtain the particular demand expression that we need to analyze the equilibrium existence:[5]

$$D_x = \begin{cases} 1, & p_x - p_y \in R^{-\infty} \\ \frac{q}{2} - \frac{a}{2b} - \frac{p_x - p_y}{2bz}, & p_x - p_y \in R_2^L \\ \frac{q}{2} - \frac{p_x - p_y}{2(a+bz)}, & p_x - p_y \in R^C \\ \frac{q}{2} + \frac{a}{2b} - \frac{p_x - p_y}{2bz}, & p_x - p_y \in R_1^L \\ 0, & p_x - p_y \in R^{+\infty} \end{cases} \tag{4}$$

In this case, $R^{-\infty} = \left[-\infty, -z\left(a + b(2-q)\right)\right]$, $R_2^L = \left[-z\left(a+b(2-q)\right), -z(a+bz)\right]$, $R^C = \left[-z(a+bz), z(a+bz)\right]$, $R_1^L = \left[z(a+bz), z(a+bq)\right]$ and $R^{+\infty} = \left[z(a+bq), +\infty\right]$, and where $z = (y-x) \in [0,1]$, i. e. is the distance between the firms, and $q = x + y \in [0,2]$ i. e. is the sum of the two locations.

Interestingly, q have a particular meaning, since $\frac{q}{2}$ represents the mean point between the locations of the firms. Furthermore, q give us the market share of the firms at equilibrium. As we will see, when $q > 1$, the demand of firm *X* is greater than the demand of firm *Y*, and *viceversa*. If $q = 1$ both firms have the same market share. On the other hand, z is a measure of the differentiation degree in the market.

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[4] Arguedas and Hamoudi (2008) analyze the concave case.

[5] We can easily deduce the demand of firm *Y* from $D_y = 1 - D_x$.

Technically, $q=1$ implies that both firms are symmetrically located with respect to the centre of the city, $q<1$ implies that both firms are asymmetrically located to the left, and $q>1$ implies that both firms are asymmetrically located to the right. This change of variable allow us to focus on the analysis of the price equilibrium for $q \geq 1$, since the demand function of firm *X* given by expression (4) is symmetric to the demand function of firm *Y* with respect to $q=1$. This fact implies symmetry of the profit functions of both firms.

As we mentioned above, without loss of generality, we consider that the production cost of both firms are equal to zero, being the profit function $\pi_s(p_s, p_r) = p_s D_s(p_s, p_r)$, where $D_s(p_s, p_r)$ is the demand function of firm S, $S = X, Y$, $s = x, y$, $r = x, y$ and $s \neq r$.

***Definition* 1:** *For a given firms location* (x, y), *a Nash-price equilibrium is the pair* $\left(p_x^N, p_y^N\right)$ *such that:*

(i) $p_x^N - p_y^N \in R^N$ *where* $R^N = R_2^L, R^C, R_1^L$.

(ii) $p_s^N = \arg\max_{p_s} \pi_s\left(p_s, p_r^N\right)$ *for all* $s = x, y$, $r = x, y$ *and* $s \neq r$.

As Anderson (1988) shows, the profit functions are piecewise concave and continuous when $a, b > 0$. In this case it is not guarantee the existence of price equilibrium for all the possible locations of the firms. Thus, it is necessary to find the conditions on firms' locations for the equilibrium existence.

4.1. Price equilibrium in the central region, R^C

We analyze the case when the indifferent consumer is located in $[x, y]$. This implies to analyze the central region of the demand function given by (3), that is, for a price difference such that $-z(a+bz) \leq p_x - p_y \leq z(a+bz)$. We denote by $(p_x^{N_C},\ p_y^{N_C})$ the candidate for a Nash price equilibrium.

According to Definition 1, if the pair $\left(p_x^{N_C}, p_y^{N_C}\right)$ is a Nash-price equilibrium then the price difference must belong to the appropriate range. In addition, each price must be the best response of the corresponding firm to the price of the other company.

In order to determine the equilibrium conditions, we first look for the necessary condition to guarantee the point (i) of Definition 1. This is given by the following result:

***Lemma* 3:** *For the existence of Nash price equilibrium in the region R^C it is necessary that $(z,q)\in E_1^C$, where $E_1^C = \left\{(z,q)\in[0,1]\times(1,2] \big/ z\geq \frac{2}{3}(q-1),\ z\geq -\frac{2}{3}(q-1)\right\}$.*

Proof:

We denote $R_s^C = \left\{p_s \big/ p_r^{N_C} - z(a+bz) \leq p_s \leq p_r^{N_C} + z(a+bz)\right\}$, $s = x, y;\ r = x, y$ and $s \neq r$.

Assuming that $\left(p_x^{N_C}, p_y^{N_C}\right)$ is a price equilibrium where:

$$p_x^{N_C} = \underset{p_x\in R_x^C}{\arg\max}\, \pi_x\left(p_x, p_y^{N_C}\right) = \frac{1}{3}(a+bz)(2+q) \tag{5}$$

$$p_y^{N_C} = \underset{p_y\in R_y^C}{\arg\max}\, \pi_y\left(p_x^{N_C}, p_y\right) = \frac{1}{3}(a+bz)(4-q) \tag{6}$$

we have:

$-z(a+bz)\leq p_x^{N_C} - p_y^{N_C} \Leftrightarrow z\geq -\frac{2}{3}(q-1)$ and $p_x^{N_C} - p_y^{N_C} \leq z(a+bz) \Leftrightarrow z\geq -\frac{2}{3}(q-1)$.■

Note that the difference in prices crucially depends on the degree of asymmetry of the locations, since $p_x^{N_C} - p_y^{N_C} = \frac{2}{3}(a+bz)(q-1)$. Therefore, $p_x^{N_C} = p_y^{N_C}$ if and only if $q=1$.[6]

We now study the point (ii) of Definition 1. This allows us determining the equilibrium conditions given by the following two lemmas:

[6] This corresponds to the case analyzed in Gabszewicz and Thisse (1986).

***Lemma* 4:** *Given* $p_y^{N_C}$ *, the profit function of firm X is decreasing in* R_1^L*.*

Proof:

Here, we denote $R_{1,x}^L = \left\{ p_x \big/ p_y^{N_C} - z(a+bz) \le p_x \le p_y^{N_C} + z(a+bz) \right\}$. Using the expression of the profit function of firm *X* on the region R_1^L, we obtain that

$$p_{x1} = \underset{p_x \in R_{1,x}^L}{\arg\max}\, \pi_x\left(p_x, p_y^{N_C} \right) = \frac{1}{6}\left(z[2b(2+q)+3a] + a(4-q) \right),$$

and the indifferent consumer $\alpha\left(p_{x1}, p_y^{N_C} \right) \ge x$. Thus p_{x1} does not belong to the interior of R_1^L, so that we have $p_y^{N_C} + z(a+bz) \ge p_{x1}$. Thus, $\pi_x\left(p_x, p_y^{N_C} \right)$ is decreasing in $R_{1,x}^L$.■

***Lemma* 5:** $p_x^{N_C}$ *is the global maximum of the profit function of firm X is and only if* $(z,q) \in E_{31}^C \cup E_{32}^C$*, where*

$E_{31}^C = \left\{ (z,q) \in E_{21}^C \cap E_{22}^C \big/ 3z^2(4b(q+2)-3a) + 2z(4b(q+2)(q-1)+3a(4-q)) - a(4-q)^2 \ge 0 \right\}$

$E_{32}^C = \left\{ (z,q) \in \bar{E}_{21}^C \big/ z(b(q+2)^2 - 12b(q-1)+18a) - a(6(4-q)-(q+2)^2 \ge 0 \right\}$,

$E_{21}^C = \left\{ (z,q) \in E_1^C \big/ a(4-q) - z(2b(4-q)+3a) \ge 0 \right\}$, $\bar{E}_{21}^C$*: complementary set of* E_{21}^C

$E_{22}^C = \left\{ (z,q) \in E_1^C \big/ -6bz^2 - z(4b(q-1)+3a) + a(4-q) \ge 0 \right\}$

Proof:

We assume that $\left(p_x^{N_C}, p_y^{N_C} \right)$ given by equations (5) and (6) is a price equilibrium. In this case, we know by Lemma 3 that $(z,q) \in E_1^C$. From Lemma 4 three cases may arise. In the first one (Figure 2), the best reply $p_{x2} = \underset{p_x \in R_{2,x}^L}{\arg\max}\, \pi_x\left(p_x, p_y^{N_C} \right) = \frac{1}{6}\left(z[2b(2+q)-3a] + a(4-q) \right)$ of firm *X* in R_2^L does belong to the interior of R_2^L and must therefore satisfy

$p_y^{N_C} - z(a + b(2-q)) \le p_{x2} \Leftrightarrow (z,q) \in E_{21}^C$ and $p_{x2} \le p_y^{N_C} - z(a + bz) \Leftrightarrow (z,q) \in E_{22}^C$. In this situation, we have $\pi_x\left(p_x^{N_C}, p_y^{N_C}\right) \ge \pi_x\left(p_{x2}, p_y^{N_C}\right) \Leftrightarrow (z,q) \in E_{31}^C$.

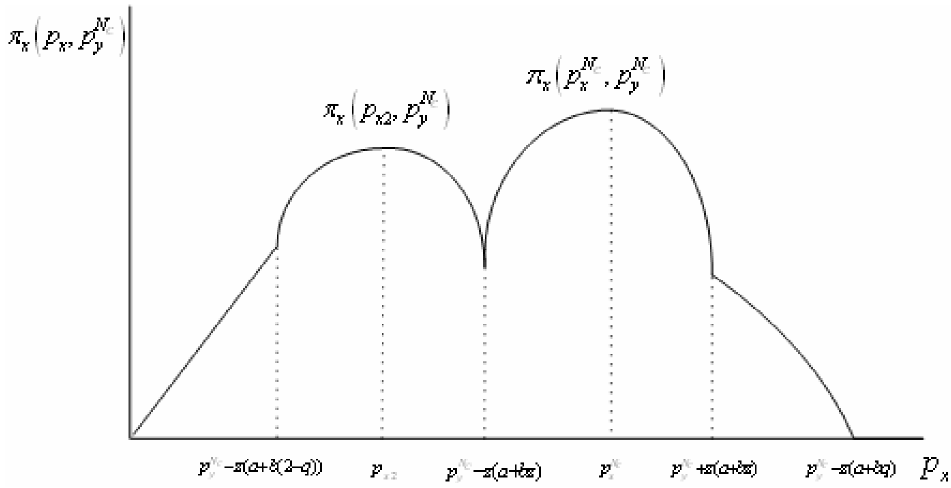

Figure 2.

In the second case (Figure 3) the local maximum in R_2^L is achieved in $p_{x2} = p_y^{N_C} - z(a + b(2-q))$ (E_{21}^C is not satisfied, i. e. $(z,q) \in \overline{E}_{21}^C$). In this situation we must verify that $\pi_x\left(p_x^{N_C}, p_y^{N_C}\right) \ge \pi_x\left(p_{x2}, p_y^{N_C}\right) \Leftrightarrow (z,q) \in E_{32}^C$.

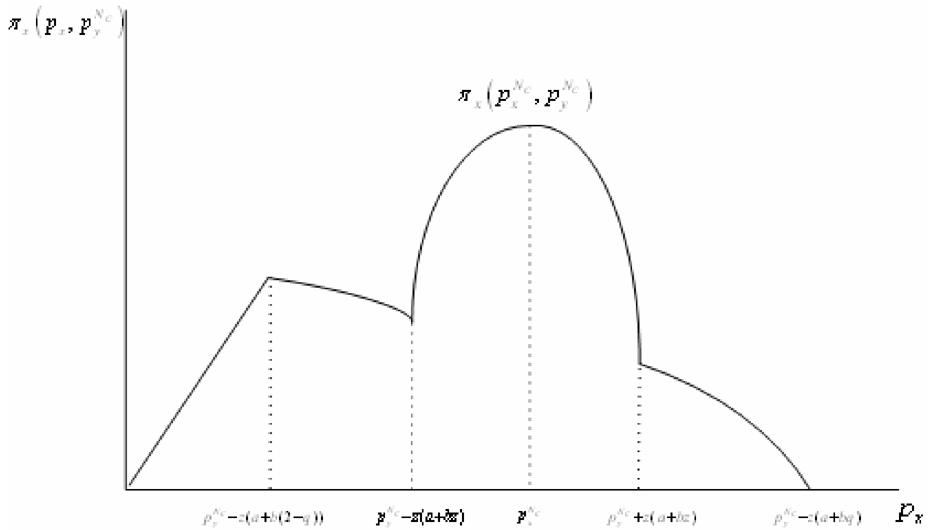

Figure 3.

In the third case (Figure 4) the local maximum in R_2^L is achieved in $p_{x2} = p_y^{N_C} - z(a+bz)$ (E_{22}^C is not satisfied, i. e. $(z,q) \in \overline{E}_{22}^C$). In this situation, w always have that $\pi_x\left(p_x^{N_C}, p_y^{N_C}\right) \geq \pi_x\left(p_x, p_y^{N_C}\right)$.

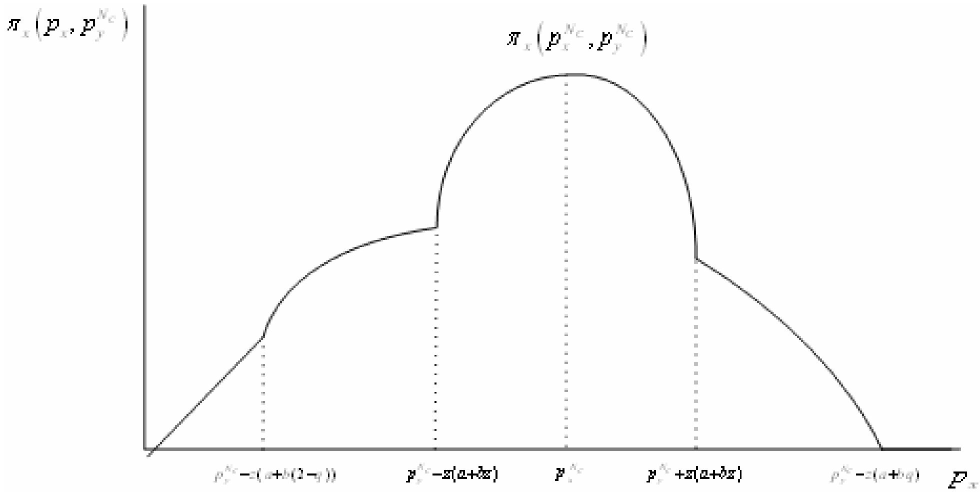

Figure 4.

■

We now have enough tools to prove our first major result:

Proposition 1: *For any* (*z, q*) *such that* $z \in [0,1]$, $q \in [1,2]$, *there is a Nash price equilibrium with* $x < \alpha < y$, *if and only if* $(z,q) \in E_{31}^C \cup E_{32}^C$, *and whenever it exist, a pure equilibrium is uniquely determined by* $\left(p_x^{N_C}, p_y^{N_C}\right)$ *given by* (5) *and* (6).

Proof:

From Lemma 5, we know that given $p_y^{N_C}$, $\pi_x\left(p_x^{N_C}, p_y^{N_C}\right) \geq \pi_x\left(p_x, p_y^{N_C}\right)$ if and only if $(z,q) \in E_{31}^C \cup E_{32}^C$. From symmetry between the profit function of firm *X* and the profit function of firm *Y*, we obtain that $\pi_y\left(p_x^{N_C}, p_y^{N_C}\right) \geq \pi_y\left(p_x^{N_C}, p_y\right)$ if and only if $(z,q) \in E_{31}^{'C} \cup E_{32}^{'C}$, where $E_{31}^{'C}$ and $E_{32}^{'C}$ are respectively the symmetric versions of E_{31}^C and E_{32}^C with respect to $q=1$, and for $q \in (1,2]$, we have that $\left(E_{31}^C \cup E_{32}^C\right) \cap \left(E_{31}^{'C} \cup E_{32}^{'C}\right) = \left(E_{31}^C \cup E_{32}^C\right)$. ■

From this result, we can easily compute the market share of the firms from the equilibrium prices given by (5) and (6). Since q >1, we see that the price and the market share of firm *X* are greater than the corresponding for the firm *Y*. For q < 1, we will find the opposite result.

423. Price equilibrium in the lateral regions

Now we study the case when the indifferent consumer is α_1^L or α_2^L (i. e. she is located in one of the lateral region). Starting from the demand function given by (4), we will search the global maximum of the profit function of both firms.

With respect to the lateral region R_2^L we have the following result:

***Lemma* 6:** For any pair $(z,q)\in(0,1]\times(1,2]$ such that z > 1- *q*, there can be no price equilibrium with $\alpha_2^L > y$.

Proof:

See proof of Lemma 4 of Andersen (1988), using *x* = (*q* - *z*)/2 and *y* = (*q* + *z*)/2. ■

In the case of the lateral region R_1^L, the necessary condition for the fulfilment of the first part of Definition 1 is given by the following lemma:

***Lemma* 7:** *For the existence of Nash price equilibrium in the region* R_1^L *it is necessary that* $(z,q)\in E_1^L$, *where* $E_1^L=\left\{(z,q)\in\left[0,1]\times[1,2]/2b(q-1)-3bz-a\geq 0\right]\right\}$.

Proof:

We now denote $R_{1,s}^C=\left\{p_s\big/p_r^{N_L}-z(a+bz)\leq p_s\leq p_r^{N_L}+z(a+bz)\right\}$, $s=x,y;\ r=x,y$ and $s\neq r$. Using the expression of the profit function in the region R_1^L, some simple calculation show that:

$$p_x^{N_L}=\arg\max_{p_x\in R_{1,x}^L}\pi_x\left(p_x,p_y^{N_L}\right)=\frac{1}{3}z\left(a+b(2+q)\right)\qquad(7)$$

$$p_y^{N_L} = \arg\max_{p_y \in R_{1,y}^L} \pi_y\left(p_x^{N_L}, p_y\right) = \frac{1}{3} z\left(a + b(4-q)\right) \tag{8}$$

In this case, the indifferent consumer is $\alpha_1^{N_L} = \dfrac{a+b(2+q)}{6b}$ and must verify $\alpha_1^{N_L} \le x$. This condition can hold if and only if $2b(q-1) - 3bz - a \ge 0$.■

Note that, for fixed locations, q=1, there exist no pure-strategy price equilibrium, since $\alpha_1^{N_L} > x$.

***Lemma* 8:** *Given $p_y^{N_L}$, the profit function of firm X is increasing in R_2^L.*

Proof:

The solution p_{x2} of the first-order conditions, $\dfrac{\partial \pi_x\left(p_x, p_y^{N_L}\right)}{\partial p_x} = 0$, given by $p_{x2} = \dfrac{1}{3} z\left(b(2+q) - 2a\right)$, is such that the indifferent consumer $\alpha_2^L \le y$. Thus, p_{x2} does not belong to interior of R_2^L. Consequently, $\pi_x\left(p_x, p_y^{N_L}\right)$ is increasing in R_2^L. ■

***Lemma* 9:** *Given $p_y^{N_L}$, it is never possible to have the profit function of firm X strictly decreasing over R^C.*

Proof:

Here $R_x^C = \left\{ p_x / p_y^{N_L} - z(a+bz) \le p_x \le p_y^{N_L} + z(a+bz) \right\}$. The maximum of $\pi_x\left(p_x, p_y^{N_L}\right)$ over R^C is reached at $p_x^C = \arg\max_{p_x \in R_x^C} \pi_x\left(p_x, p_y^{N_L}\right) = \dfrac{a}{3}\left(a(3q - z) + 2bz(2+q)\right)$. In this situation, we have always $p_y^{N_L} - z(a+bz) \le p_x^C$. For any given $z \in [0,1]$, $q \in [1,2]$ it means that $\pi_x\left(p_x, p_y^{N_L}\right)$ is not decreasing in R^C. ■

***Lemma 10*:** *The optimum of $\pi_x\left(p_x, p_y^{N_L}\right)$ over $[0, \infty]$ is reached at $p_x^{N_L}$ if and only if $(z,q) \in E_3^L \cup \overline{E}_2^L$, where:*

$E_3^L = \left\{(z,q) \in E_2^L \,/\, 3bz^2(4bq + a + 8b) + z(2bq(7a - 4bq - 4b) + 4(a+2b)^2 - 9abq^2 \geq 0\right\}$

$E_2^L = \left\{(z,q) \in E_1^L \,/\, a(z+3q)4bz(q-1) - 6z(a+bz) \leq 0\right\}$, $\bar{E}_2^L$ *: complementary set of* E_2^L

$E_1^L = \left\{(z,q) \in [0,1] \times [1,2] \big/ 2b(q-1) - 3bz - a \geq 0\right\}$

Proof:

We assume that $\left(p_x^{N_L}, p_y^{N_L}\right)$ given by expressions (7) and (8) is a price equilibrium. In this case, we know by Lemma 7 that $(z,q) \in E_1^L$. From Lemma 8 and 9, two cases may arise. In the first one (Figure 5), the best reply p_x^C of firm *X* in R^C does belong to the interior of R^C and must therefore satisfy $p_y^{N_L} - z(a+bz) \leq p_x^C \leq p_y^{N_L} + z(a+bz) \Leftrightarrow (z,q) \in E_2^L$. In this situation, we have $\pi_x\left(p_x^{N_L}, p_y^{N_L}\right) \geq \pi_x\left(p_x^C, p_y^{N_L}\right) \Leftrightarrow (z,q) \in E_3^L$.

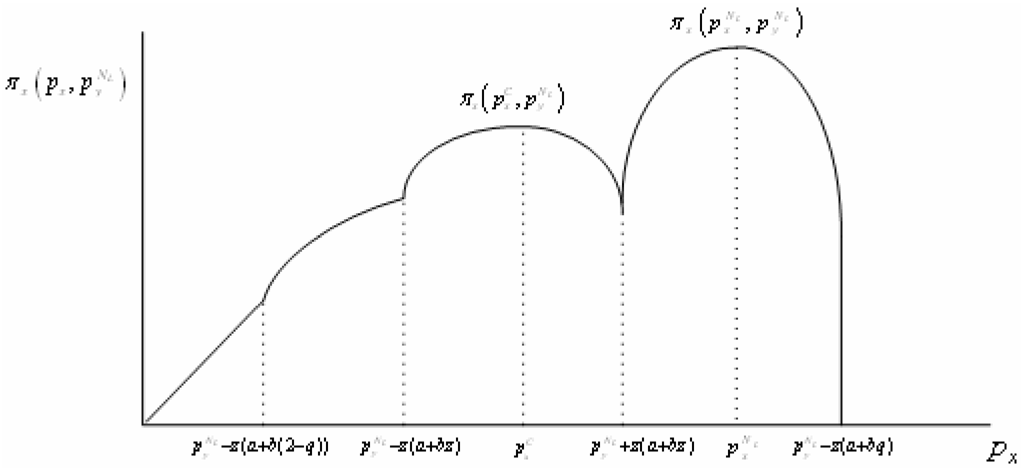

Figure 5.

In the second case (Figure 6), the profit function of firm *X* is increasing in R^C and we have $p_x^C \leq p_y^{N_L} - z(a+bz) \Leftrightarrow (z,q) \in \bar{E}_2^L$, $\pi_x\left(p_x^{N_L}, p_y^{N_L}\right) \geq \pi_x\left(p_x, p_y^{N_L}\right)$ for any given p_x.

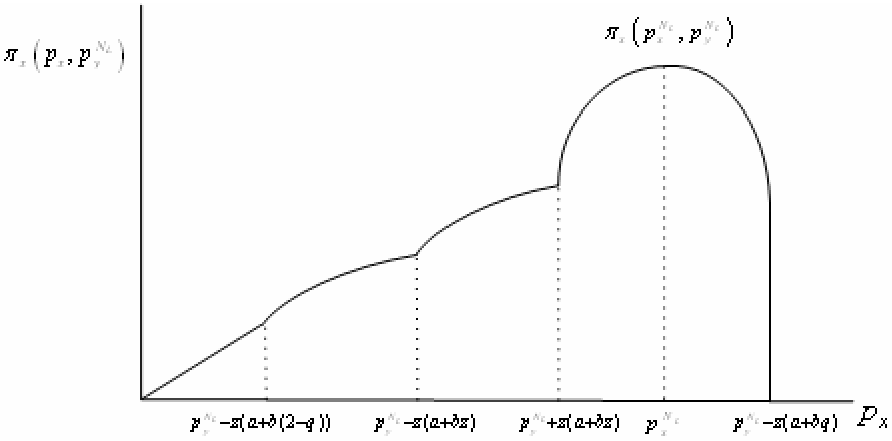

Figure 6.

■

***Proposition* 2:** For any (*z*, *q*) such that $z \in [0,1]$, $q \in [1,2]$, there is a Nash price equilibrium with $\alpha < x$ if and only if $(z,q) \in E_3^{'L} \cup \overline{E}_2^{'L}$, and whenever it exist, a pure equilibrium is uniquely determined by $\left(p_x^{N_L}, p_y^{N_L}\right)$.

Proof:

From Lemma 10, we know that given $p_y^{N_L}$, $\pi_x\left(p_x^{N_L}, p_y^{N_L}\right) \geq \pi_x\left(p_x, p_y^{N_L}\right)$ if and only if $(z,q) \in E_3^{L} \cup \overline{E}_2^{L}$. We now turn to the profit function of firm *Y*. From symmetry between the profit function of firm *X* and the profit function of firm *Y*, we obtain that $\pi_y\left(p_x^{N_L}, p_y^{N_L}\right) \geq \pi_y\left(p_x^{N_L}, p_y\right)$ if and only if $(z,q) \in E_3^{'L} \cup \overline{E}_2^{'L}$, where $E_3^{'L}$ and $\overline{E}_2^{'L}$ are respectively the symmetric versions of E_3^{L} and $\overline{E}_2^{L}$ with respect to *q* = 1, and for $q \in (1,2]$, we have that $\left(E_3^{L} \cup \overline{E}_2^{L}\right) \cap \left(E_3^{'L} \cup \overline{E}_2^{'L}\right) = \left(E_3^{'L} \cup \overline{E}_2^{'L}\right)$.■

5. Conclusions

In this paper we have studied the existence of the sequential equilibrium in the context of the traditional Hotelling model of spatial competition. First, we have studied the demand function structure using a generic transport cost function which generalizes at the same time the convex and the concave case. We find that the demand function is always connected in the convex case whereas in the concave case this function can be non-connected

In the equilibrium study, we analyze the class of linear-quadratic and convex transport cost function. We propose a decisive change of variable that permit that the profit functions of both firms being symmetric to respect to the mean point between the locations of the firms. This allows us to simplify the analysis, and characterize the exact regions of location pairs for which a price equilibrium exist in a general framework. In this sense, our general result closes the analysis on the existence of equilibrium for the class of linear-quadratic and convex transport cost function.

Appendix

The following figures correspond to the equilibrium regions for q > 1. For q < 1 the equilibrium region are symmetric with respect to q=1.

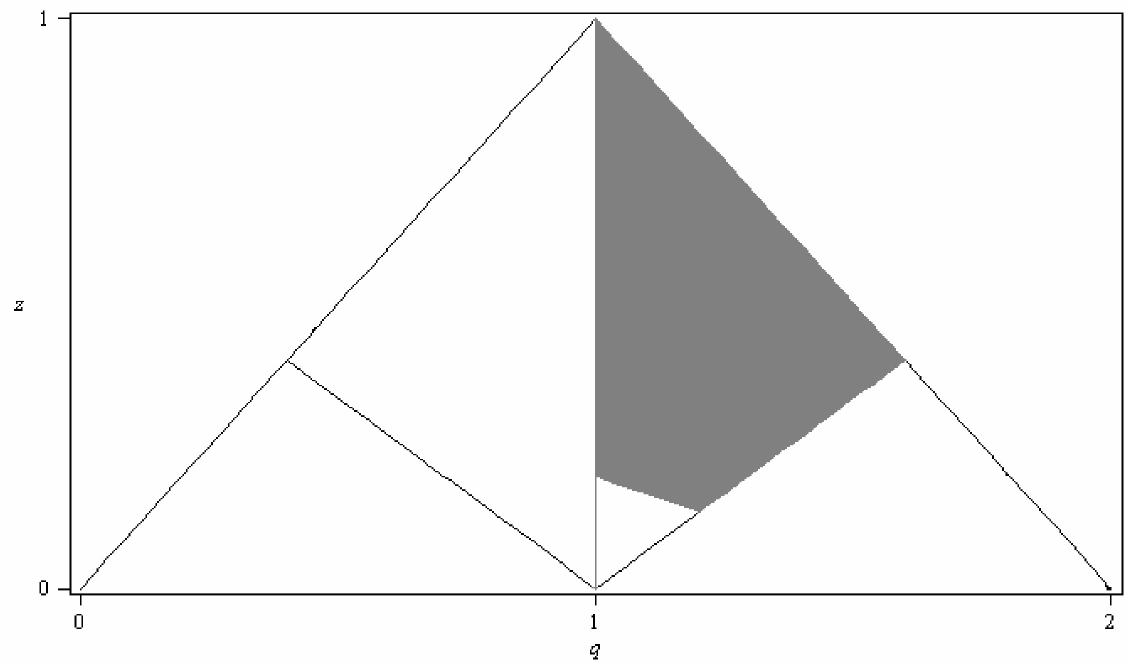

Figure 7. Equilibrium in the central region when a=1, b=4

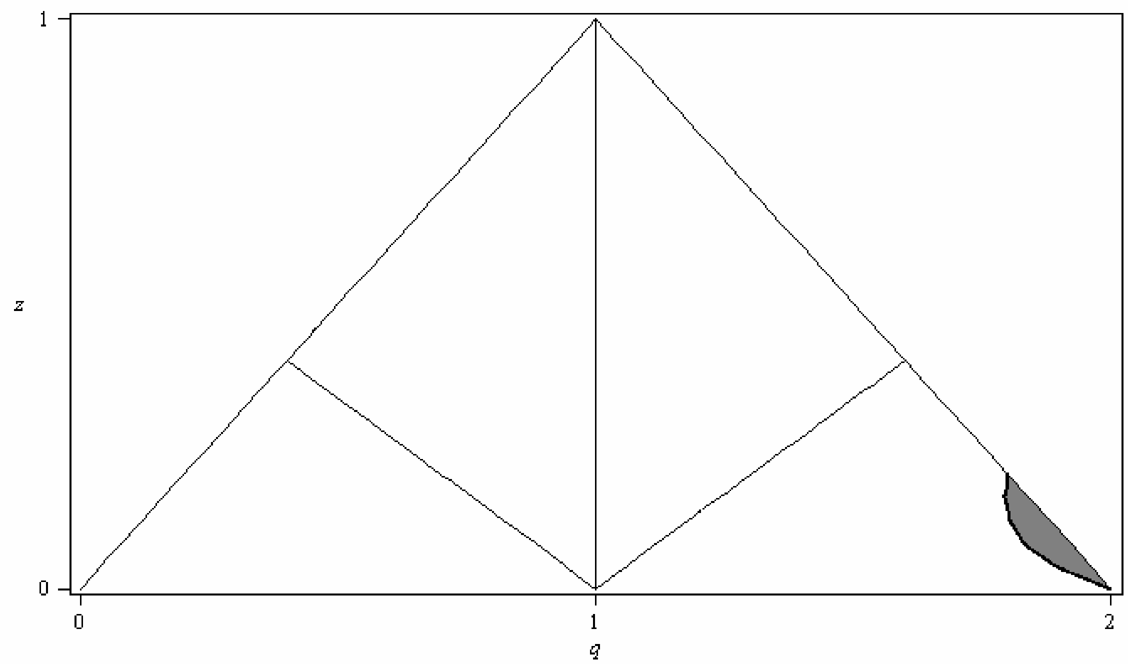

Figure 8. Equilibrium in the lateral 1 region when a=1 b=4

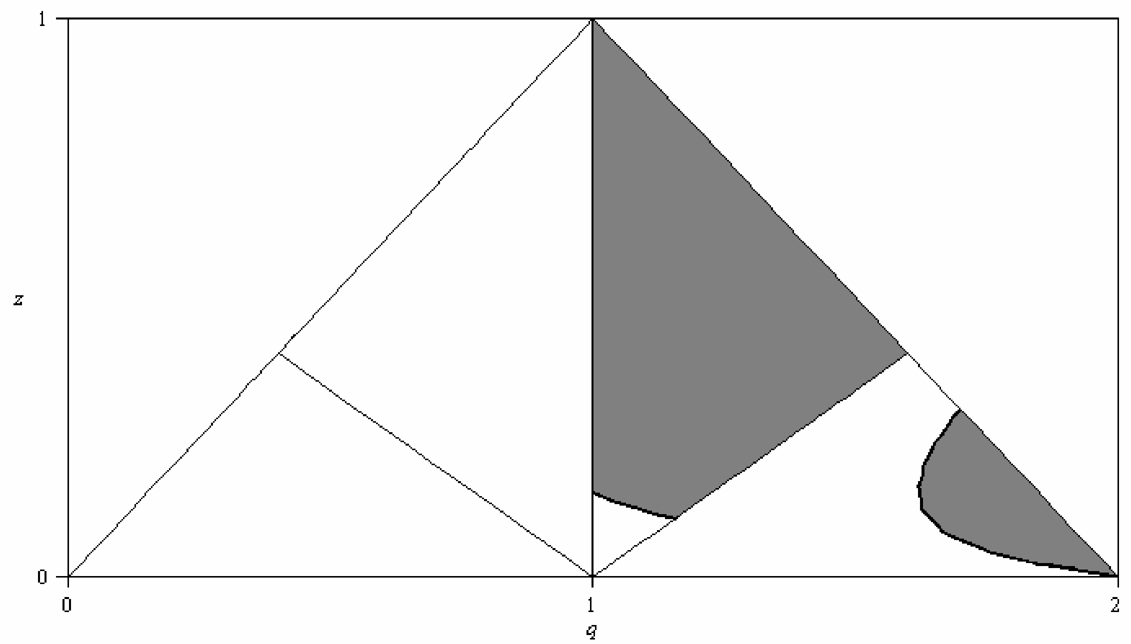

Figure 9. Equilibrium regions when a = 1, b = 8

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