Financing Constraints and Firm Dynamics with Durable Capital*

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Abstract

We consider a dynamic financing problem of a firm subject to moral hazard problems. With respect to the existing literature (e.g. as Clementi and Hopenhayn [2] and Quadrini [12]), we enrich the model by introducing durable capital and a stochastic liquidation value. The existence of durable capital allows us to make predictions based on the firm size, independently of age, while the stochastic liquidation value makes it possible to have liquidation with positive probability under the first best. We find that a higher level of capital decreases the probability of liquidation, increases future size and reduces the average return and volatility of the firm. Also, under certain parameter values a stochastic liquidation value makes it possible to achieve the first best. The results are broadly in agreement with the empirical results on the effects of firm size.

1 Introduction

This paper considers the problem of designing an optimal financial contract between an entrepreneur and a financier in a multi–period framework. The empirical literature has highlighted that the investment choices of firms frequently depend on the availability of internal funds (see Hubbard [8] for a survey). Moreover, as emphasized by Cooley and Quadrini [3], this dependence diminishes with both age and size. These results are usually explained by the existence of financial constraints for the firms, constraints which appear independently of their investment opportunities.

Recent theoretical models have explored the causes and consequences of financing constraints. Cooley and Quadrini [3], for instance, show that exogenous borrowing constraints and persistent shocks can generate the kind of relationship between age and growth of the firm that we find in the empirical data. Similarly, Lorenzoni and Walentin [9] analyze

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a model with limited enforcement in which a continuum of firms are subject to aggregate economy-wide shocks and can issue state-dependent securities. However they do not analyze the optimal multi-period financial contract that may result from bilateral negotiations.

Other research has analyzed the design of optimal dynamic financial contracts, trying to obtain financing constraints as part of an optimal contract. Borrowing constraints can appear endogenously because of information asymmetry or limited enforcement of contracts, and we focus on the first case[1]. The main theme of this literature is that financing constraints are an optimal way to provide incentives to the borrower. The incentives take the form either of a threat to transfer control (including liquidation), as in Kiyotaki and Moore [10] and DeMarzo and Fishman [5], or a threat to reduce future financing (see e.g. Gromb [6]). Our model builds on the work of Quadrini [12] and, more closely, Clementi and Hopenhayn [2], where both types of threat are used. Their emphasis is on the importance of financing constraints as determinants of firm dynamics, especially size, growth and survival. They develop a theory of endogenous financing constraints as part of the optimal design of a lending contract under asymmetric information and they show that in an optimal financial contract the amount of investment and the value of equity increase with high revenue shocks and decrease with low ones. The sensitivity of equity value to revenue shocks provides incentives to the entrepreneur to reveal the true value of the shock. Financing constraints tend to disappear when the value of equity becomes sufficiently large, which in turn happens as the firm approaches the optimal size.

We enrich the model in two ways. First, we assume that capital is durable. In our model capital depreciates at a rate $1 - d$, with $d \in [0, 1]$, and it can be augmented each period by new investment or diminished by selling part of the existing capital. In the Clementi-Hopenhayn model, the working capital invested at period t depreciates completely at the end of the period, that is $d = 0$. Introducing durable capital is useful because it allows us to analyze how the optimal contracting problem changes as the size of the firm changes. Since size is now an independent state variable, we can analyze its effect on the probability of liquidation and investment independently from other variables, such as age or equity value. In contrast, in Clementi-Hopenhayn size has to be decided in every period as the amount of working capital invested in the firm, so that the incentive problem remains stationary. This implies that the probability of liquidation and future investment depend on the current size only through the current value of equity.

Second, Clementi and Hopenhayn [2] assume that the liquidation value of the firm is constant, while we allow for a stochastic liquidation value. Our hypothesis is that the liquidation value is higher after the firm has received a positive revenue shock; this assumption can be justified assuming that liquidation is attained selling the firm's assets to other firms in the industry, and that revenue shocks for the firm are positively correlated

[1] See Albuquerque and Hopenhayn [1] for a model of optimal financial contracts and firm dynamics with limited enforcement. Cooley, Marimon and Quadrini [4] also consider a model with limited enforcement and optimal financial contracts, and they assume that a defaulting entrepreneur can have a fresh start in the next period.

to industry shocks (see e.g. Shleifer and Vishny [14]).

The results can be summarized as follows. First, when the liquidation value is stochastic it is possible, depending on the value of the parameters, that liquidation occurs with positive probability in equilibrium. More specifically, liquidation may occur after a good shock that raises the value of capital. In such cases, since the liquidation value is observable, moral hazard becomes less of a problem. In fact, there are values of the parameters for which the first best becomes implementable, something that cannot happen with non-stochastic liquidation values.

Second, in general the optimal second-best policy may prescribe inefficient liquidation and underinvestment. However, these effects are less marked for firms of bigger size. A higher level of capital decreases the optimal probability of liquidation and increases the future level of capital. These effects are independent and separate from the effect of an increase in the value of equity, discussed in Clementi and Hopenhayn [2] and Quadrini [12]. We also find that firms of bigger size, other things equal, have a return on assets with both lower expected value and lower volatility. Since the returns on the firm's assets are defined as part of a bilateral contract, this is not the result of an equilibrium trade-off between risk and return determined in the financial markets; our agents are risk neutral, and in the absence of incentive problems they would push investment to the point at which the expected return equals the risk–free rate. Rather, a higher rate of return is observed only when the firm is implementing a sub-optimal policy for incentive reasons; higher levels of capital are associated to less inefficient policies, which in turn yield lower expected returns.

The rest of the paper is structured as follows. Section 2 introduces the model and analyzes the optimal investment policy when there are no agency problems. In section 3 we start the analysis of the optimal financial contract under asymmetric information. The analysis is continued in section 4, where we discuss the liquidation policy, and in section 5 where we analyze the impact of size on the optimal policy. Section 6 contains the conclusions. An appendix contains the proofs.

2 The Model

At time 0 an entrepreneur has an idea for a project. Implementing the project requires forming a firm and acquiring an enabling asset at cost A. After the project has been activated, the firm needs capital to operate. The cash flow at time t depends on the amount of capital K_t existing at that time and on a random variable $\widetilde{\theta}_t$. Let

$$\widetilde{\theta}^t = \left(\widetilde{\theta}_1, \widetilde{\theta}_2, \ldots, \widetilde{\theta}_t\right)$$

be the history of shocks up to time t. Realizations of $\widetilde{\theta}_t$ and $\widetilde{\theta}^t$ will be denoted by θ_t and θ^t respectively. At the beginning of each period t the project can be continued or liquidated. If it continues, then the firm makes an investment decision and the cash flow produced by the project is $\theta_t R(K_t)$. If it is liquidated, then the existing assets are sold at a value

$S(\theta_{t-1}, K_{t-1})$, that we will discuss shortly. We make the following assumption about the production process.

Assumption 1 *The random variables* $\left\{\widetilde{\theta}_t\right\}_{t=1}^{+\infty}$ *are independent and identically distributed, with* $\Pr\left(\widetilde{\theta}_t = 1\right) = p$ *and* $\Pr\left(\widetilde{\theta}_t = 0\right) = 1 - p$*. The function* $R(\cdot)$ *is defined on* $[0, +\infty)$*. It is bounded, continuously differentiable, strictly increasing, strictly concave and it satisfies* $R(0) = 0$ *and* $pR'(0) > 1$*.*

Capital depreciates at a rate $1-d$, and it can be increased by new investment or decreased by selling part of the existing capital. Denoting ΔK_t as the change in capital level through investment or sale at time t, the law of motion of capital becomes

$$K_t = dK_{t-1} + \Delta K_t,$$

with initial level of capital $K_0 = 0$. With respect to the liquidation value $S(\theta_{t-1}, K_{t-1})$, we make the following assumption.

Assumption 2 *The liquidation value depends linearly on capital:*

$$S(\theta_{t-1}, K_{t-1}) = S(\theta_{t-1}) + q(\theta_{t-1})\, dK_{t-1},$$

with $0 \leq S(0) \leq S(1)$ *and* $0 \leq q(0) \leq q(1) < 1$*.*

To simplify notation we will set $S(0) = \underline{S}$, $S(1) = \overline{S}$, $q(0) = \underline{q}$ and $q(1) = \overline{q}$. The interpretation here is that $S(\theta_{t-1})$ is the liquidation value obtained selling the enabling asset, while $q(\theta_{t-1})$ is the unit price at which capital assets can be sold. Capital assets have use outside the firm, so that $\underline{q} \geq 0$, but the productivity of capital assets is not as high outside the firm as it is inside, hence the condition $\overline{q} < 1$. Notice that $\overline{q} < 1$ implies that it can never be optimal to increase the amount of capital only to resell it in the future. The assumption that the liquidation value is higher after a good shock than after a bad shock can be justified assuming that the firm's shock is correlated to the shocks of other firms in the industry. Shleifer and Vishny [14] explain this effect focusing on the potential buyers of the assets of a firm. The assets of a liquidated firm can be sold to other firms in the industry, which are the best potential buyers (they can give the assets their best use). But, if a firm suffers financial distress, its industry peers are likely to be experiencing problems themselves. Therefore, they are less likely to buy the assets, which are therefore sold to industry outsiders with a lower valuation.

Let $\delta \in (0, 1)$ be the discount factor at which all agents discount future cash flows. In order to make the problem interesting, we will make the following assumption.

Assumption 3 $A > \delta\left[p\overline{S} + (1-p)\underline{S}\right]$*.*

The assumption implies that it cannot be optimal to buy the enabling asset only to liquidate the firm in the following period. If the firm is set up, some nontrivial investment policy is necessary in order to make the enterprise profitable.

The entrepreneur is protected by limited liability, meaning that the monetary transfer from the firm to the lender in period t cannot exceed $\theta_t R(K_t)$. Furthermore, we assume that the cash flow at time t is not verifiable, and it is private information of the entrepreneur. This means that the entrepreneur can hide the whole income of the firm and divert it to her own accounts. The financing contract will have to provide incentives to the entrepreneur to report correctly the firm's cash flow, which is equivalent to reporting correctly the value θ_t.

2.1 Feasible Contracts

The entrepreneur has insufficient funds to start the project, and therefore needs financing from a lender. A financing contract specifies the sum M_L to be given by the lender to the borrower at time zero, the amount of cash M_B that the borrower has to invest in the project, future investments and future payments between the two parties as a function of history. At $t = 0$ the only possible activity is the acquisition of the enabling asset at price A. Furthermore, the realization of the random variable $\widetilde{\theta}_0$ is observed by the entrepreneur. From period 1 on the firm can start investing in capital and producing.

At each period $t \geq 1$ at which the firm is active the sequence of events is the following. First, a liquidation decision $\ell_t \in \{L, C\}$ is taken, where $\ell_t = L$ means liquidation and $\ell_t = C$ means continuation. Whether or not a liquidation decision is taken is specified by the contract. More generally, for any given history θ^{t-1} the contract specifies the probability $\alpha_t = \Pr(\ell_t = L)$, where $\alpha_t \in [0, 1]$ depends on history.

If the firm is liquidated then the contract is over, and the scrap value $S_t = S(\theta_{t-1}, K_{t-1})$ is distributed between the lender and the borrower according to the terms defined by the contract. We call Q_t the amount paid to the borrower in case of liquidation at time t; the lender obtains $S_t - Q_t$. If $\ell_t = C$ then the contract specifies the amount of capital K_t for period t (or, more in general, a probability distribution $\kappa_t(h_{t-1})$ over K_t). If $K_t \geq dK_{t-1}$ then new investment is required and the cost is $I_t = K_t - dK_{t-1}$. If $K_t < dK_{t-1}$ then the firm sells part of its capital, obtaining a revenue equal to $q(\theta_{t-1})(dK_{t-1} - K_t)$. Again, the amount K_t prescribed by the contract is a function of past history. For future reference, we define the cost of investment as

$$I_t(\theta_{t-1}, K_t, K_{t-1}) = \begin{cases} K_t - dK_{t-1} & \text{if} \quad K_t \geq dK_{t-1} \\ q(\theta_{t-1})(K_t - dK_{t-1}) & \text{if} \quad K_t < dK_{t-1}. \end{cases}$$

Once the new capital level is selected the outcome $\theta_t R(K_t)$ is observed by the borrower. Then the borrower sends a verifiable message m_t to the lender and pays him an amount τ_t that depends on the current message, the level of capital K_t chosen and on past history. We invoke the revelation principle and let the set of messages at time t be $\Theta = \{0, 1\}$, i.e. the entrepreneur reports the observed value of θ_t. An outcome a_t at time $t \geq 0$ is given by

$$a_t = \left\{ \ell_t, K_t, \widehat{\theta}_t \right\}$$

where $\widehat{\theta}_t$ is the message issued by the borrower. We set $\ell_0 = C$ and $K_0 = 0$, which just means that at period zero the firm is started and investment in capital starts from period one[2]. A history up to time t is a collection $h_t = \{a_s\}_{s=0}^t$ and H_t is the set of all possible histories up to time t. A *feasible financing scheme* is a collection of functions

$$f = \left\{ \alpha_t(h_{t-1}), Q_t(h_{t-1}, S_t), \kappa_t(h_{t-1}), \tau_t\left(\widehat{\theta}_t, K_t, h_{t-1}\right) \right\}_{t=1}^{+\infty}$$

with the following properties:

- For each history h_t such that $a_s = (L, \cdot, \cdot)$ for some $s \leq t$ we have

  $$Q_{t'} = \tau_{t'} = S_{t'} = K_{t'} = 0$$

  for each $t' > s$. In other words, after a liquidation decision the firm stops operating.

- For each history h_t we have $\alpha_{t+1}(h_t) \in [0,1]$ and $K_{t+1}(h_t) \geq 0$, i.e. the probability of liquidation has to be between zero and one and capital has to be non-negative.

- For each history h_t, $Q_{t+1}(h_t, S_{t+1}) \geq 0$ and $\tau_t\left(\widehat{\theta}_t, K_t, h_{t-1}\right) \leq \widehat{\theta}_t R(K_t)$. This is the consequence of borrower's limited liability.

A contract is a triplet

$$\sigma = (M_L, M_B, f)$$

where $M_L \geq 0$ is the amount of capital invested by the lender, $M_B \geq 0$ is the amount of capital invested by the borrower and f is a financing scheme. The contract is feasible if $M_L + M_B \geq A$ and f is a feasible financing scheme.

Let

$$P_t^L(h_{t-1}, \theta_t) = \alpha_t(h_{t-1})\left(S_t - Q_t(h_{t-1}, S_t)\right) + \left(1 - \alpha_t(h_{t-1})\right)\left(\tau_t(\theta_t, h_{t-1}) - I_t(h_{t-1})\right)$$

be payoff to the lender at time t, after history h_{t-1}, when $\widetilde{\theta}_t = \theta_t$ and $S_t = S_t(\theta_{t-1}, K_{t-1}(h_{t-2}))$, where $I_t(h_{t-1})$ is computed using the optimal policy $K_t(h_{t-1})$ (we are assuming that the lender received the profits in case of sale of capital and pays the cost of investment when capital is increased).

Similarly, let

$$P_t^B(h_{t-1}, \theta_t) = \alpha_t(h_{t-1}) Q_t(h_{t-1}, S_t) + \left(1 - \alpha_t(h_{t-1})\right)\left(\theta_t R(K_t(h_{t-1})) - \tau_t(\theta_t, h_{t-1})\right)$$

be the payoff to the borrower.

[2] Results don't change if we allow for some investment at $t = 0$ (besides the purchase of the enabling asset), but setting $K_0 = 0$ makes the notation simpler.

The contract is *individually rational* if

$$E\left[\sum_{t=1}^{+\infty}\delta^t P_t^L\left(h_{t-1},\widetilde{\theta}_t\right)\right]\geq M_L \tag{1}$$

$$E\left[\sum_{t=1}^{+\infty}\delta^t P_t^B\left(h_{t-1},\widetilde{\theta}_t\right)\right]\geq M_B \tag{2}$$

where the expectation is taken over all possible histories and under the assumption that the borrower reports the true value θ_t at each period.

Let $\widehat{r}_t:\Theta\times H_{t-1}\to\Theta$ be a reporting strategy at time t for the borrower and $\widehat{\mathbf{r}}=\{\widehat{r}_t\}_{t=0}^{+\infty}$ be a reporting strategy for all periods. Denote with $\mathbf{r}$ the truthtelling strategy, that is $r_t(\theta_t,h_{t-1})=\theta_t$ for each (θ_t,h_{t-1}). Define

$$V_t^{\widehat{\mathbf{r}}}(\theta_t,h_{t-1})=E\left[\sum_{q=t}^{+\infty}\delta^{q-t}\alpha_q(\widehat{r}_{q-1},h_{q-1})\,Q_q\left(h_{q-1},\widetilde{S}_q\right)\middle|\,h_{t-1}\right]$$

$$+E\left[\sum_{q=t}^{+\infty}\delta^{q-t}(1-\alpha_q(\widehat{r}_{q-1},h_{q-1}))\left(\widetilde{\theta}_qR(K_q(\widehat{r}_{q-1},h_{q-1}))-\tau_q(\widehat{r}_q,h_{q-1})\right)\middle|\,h_{t-1}\right]$$

as the present value of the expected payment to the entrepreneur from time t on, given history h_{t-1} and reporting strategy $\widehat{\mathbf{r}}$. Here we have emphasized that the contractual variables α_q and K_q at time q depend on h_{q-1} also through the last period announcement $\widehat{r}_{q-1}$, and the transfer at time q depends on the current period announcement $\widehat{r}_q$. Notice also that in general $\widehat{r}_q$ is a function of h_{q-1} and θ_q.

A contract is *incentive compatible* if

$$V_t^{\mathbf{r}}(\theta_t,h_{t-1})\geq V_t^{\widehat{\mathbf{r}}}(\theta_t,h_{t-1})$$

for each history (θ_t,h_{t-1}) and reporting strategy $\widehat{\mathbf{r}}$. Since we look at contracts inducing truth-telling, it will be convenient to simplify notation by setting $V_t(\theta_t,h_{t-1})\equiv V_t^{\mathbf{r}}(\theta_t,h_{t-1})$.

2.2 Optimal Investment under Complete Information

Before analyzing the incomplete information case, we characterize the optimal policy when there are no agency problems; this is the case, for example, when the entrepreneur has enough cash to finance entirely the acquisition of the enabling asset A. The value of the firm at the beginning of time t when the capital in previous period was K_{t-1} is defined as

$$W(\theta_{t-1},K_{t-1})=$$

$$\sup_{\{\alpha_q(h_{q-1}),K_q(h_{q-1})\}_{q=t}^{+\infty}\in\mathcal{F}} E\left[\sum_{q=t}^{+\infty}\delta^{q-t}\left(\alpha_q\widetilde{S}_q+(1-\alpha_q)\left(\widetilde{\theta}_qR(K_q)-I_q\left(\widetilde{\theta}_{q-1},K_q,K_{q-1}\right)\right)\right)\right]$$

where $\mathcal{F}$ is the set of feasible investment and liquidation policies. Since the function $I_t(\theta_{t-1}, K_t, K_{t-1})$ is strictly decreasing in K_{t-1}, it immediately follows that $W(\theta_{t-1}, \cdot)$ is increasing. Let

$$W_c(\theta_{t-1}, K_{t-1}) = \max_{K_t \geq 0} \quad pR(K_t) - I_t(\theta_{t-1}, K_t, K_{t-1}) + \delta E\left[W\left(\widetilde{\theta}_t, K_t\right)\right]$$

be the maximum value attainable by the firm when the capital is K_{t-1} and the firm does not liquidate in the current period. Notice that the value depends on θ_{t-1} only if $K_t < dK_{t-1}$ and observe that $W_c(\theta_{t-1}, K_{t-1})$ is strictly increasing in K_{t-1}. Then

$$W(\theta_{t-1}, K_{t-1}) = \max\{S(\theta_{t-1}, K_{t-1}), W_c(\theta_{t-1}, K_{t-1})\}.$$

As previously pointed out, Clementi and Hopenhayn [2] have analyzed the case of non-durable capital and non-stochastic liquidation value, i.e. $d = 0$ and $S_t(\theta_{t-1}, K_{t-1}) = S$. For that case, under complete information the optimal policy is simply to select an investment K^{CH} in every period solving

$$pR'\left(K^{CH}\right) = 1. \tag{3}$$

The value function is therefore independent of θ_t and equal to $W^{CH} = \frac{pR\left(K^{CH}\right) - K^{CH}}{1-\delta}$.

In the general case the optimal policy is more complicated. We start introducing some notation and establishing some preliminary results. Let $\widehat{W}(\theta_{t-1}, K_{t-1})$ be the value that can be achieved if the firm never liquidates the project, that is

$$\widehat{W}(\theta_{t-1}, K_{t-1}) = \max_{\{K_q\}_{q=t}^{+\infty}} \quad E\left[\sum_{q=t}^{+\infty} \delta^{q-t}\left(pR(K_q) - I_q(\theta_{q-1}, K_q, K_{q-1})\right)\right]. \tag{4}$$

Assume for the moment that the optimal policy is such that $K_t \neq dK_{t-1}$ at each t. Then I_t is differentiable and the first order condition with respect to K_t is

$$pR'(K_t) = \frac{\partial I_t}{\partial K_t} + \delta E\left[\left.\frac{\partial I_{t+1}}{\partial K_t}\right| h_{t-1}\right] \tag{5}$$

where

$$\frac{\partial I_t}{\partial K_t} = \begin{cases} 1 & \text{if} \quad K_t > dK_{t-1} \\ q(\theta_{t-1}) & \text{if} \quad K_t < dK_{t-1}. \end{cases}$$

and

$$\frac{\partial I_t}{\partial K_{t-1}} = \begin{cases} -d & \text{if} \quad K_t > dK_{t-1} \\ -q(\theta_{t-1})\,d & \text{if} \quad K_t < dK_{t-1}. \end{cases}$$

The next lemma establishes that the optimal policy prescribes positive investment for each $t \geq 1$.

Lemma 1 *Consider the problem of maximizing the value of the firm when liquidation is not allowed. Then, under the optimal policy, $K_t > dK_{t-1}$ implies $K_{t+1} > dK_t$ for each value θ_t.*

The lemma states that, when the optimal policy is never to liquidate the firm, either investment is positive on the optimal path and capital is never sold or the investment policy is trivial and always equal to zero. The intuition is that a strictly positive investment followed by negative investment can never be optimal, since $q(\theta_t) < 1$. The firm can do better by reducing investment at time t, thus avoiding overinvestment. The lemma also implies that, when the optimal policy is never to liquidate the firm and investment is positive, the level of capital is constant. Since $K_1 > 0$, the lemma implies $K_t > dK_{t-1}$ for each t. Thus, condition (5) becomes

$$pR'(K_t) = 1 - d\delta$$

for each t, implying a constant level of capital. In fact, define K^* as the solution to

$$pR'(K^*) = 1 - d\delta. \tag{6}$$

Then, when the optimal policy is never to liquidate the firm, the optimal investment policy is $K_1 = K^*$ and $I_t = (1-d)K^*$ for each $t > 1$, that is the firm immediately reaches the stationary level K^* and then simply replaces the depreciated capital in each period.

We can now show that, when the optimal policy is to have positive investment then the function $\widehat{W}(\theta, K)$ is strictly increasing in K. The function depends on θ only out of the optimal path, when K is large and it is optimal to sell part of it. The value of the firm at the beginning of time 1 is

$$\begin{aligned}\widehat{W}(\theta, 0) &= \frac{pR(K^*) - K^*}{1-\delta} + \delta\frac{dK^*}{1-\delta} \\ &= \frac{pR(K^*) - (1-d\delta)K^*}{1-\delta}.\end{aligned}$$

At time 0 the value of the project is therefore

$$\widehat{W}^* = \delta\widehat{W}(\theta, 0). \tag{7}$$

More in general, when $K < \frac{K^*}{d}$ the value of the firm is

$$\widehat{W}(\theta, K) = \widehat{W}(\theta, 0) + dK. \tag{8}$$

which is increasing in K. The only values observed on the optimal path are 0 and K^*.

2.3 Optimal Liquidation

Remember that, given the amount of capital accumulated K_{t-1}, at the beginning of every period t the value of the firm is given by

$$W(\theta_{t-1}, K_{t-1}) = \max\{S(\theta_{t-1}, K_{t-1}), W_c(\theta_{t-1}, K_{t-1})\}.$$

If the optimal policy never allows for liquidation then $W_c(\theta, K) = \widehat{W}(\theta, K)$, and the discussion above implies that the only values of capital observed under the optimal policy are 0 and K^*. No liquidation is optimal if and only if the continuation value is always higher than the liquidation value. Since for each level of capital the liquidation value is at the highest level when $\theta = 1$, a necessary condition for the optimality of the no liquidation policy is

$$\widehat{W}(\theta, 0) \geq \overline{S}. \tag{9}$$

Notice further that $S(1, K) = \overline{S} + \overline{q}dK$ and $\widehat{W}(\theta, K) = \widehat{W}(\theta, 0) + dK$ for $K \leq \frac{K^*}{d}$. Since $\overline{q} < 1$, condition (9) implies

$$\widehat{W}(\theta, K) > S(\theta, K) \tag{10}$$

for each θ and $K \leq \frac{K^*}{d}$. Therefore, condition (9) is both necessary and sufficient for no–liquidation to be optimal.

Suppose now that condition (9) is violated. It must then be the case that liquidation occurs with positive probability on the optimal path. We start observing that it can never be the case that, on the optimal path, liquidation occurs when the liquidation value is low.

Lemma 2 *It can never be the case that, along the optimal path, liquidation occurs when $\theta = 0$.*

The logic of the result is the following. Assumption 3 implies that, if the project has positive NPV, when $K = 0$ it is optimal to continue at $\theta = 0$, i.e. $W_c(0, 0) > S(0, 0)$. On the optimal path we have $W_c(0, K) > W_c(0, 0) + \underline{q}dK$, since the firm can sell the existing capital and adopt the same policy adopted when capital is zero[3]. Given that $S(0, K) = S(0, 0) + \underline{q}dK$, it follows that $W_c(0, K) > S(0, K)$ for each K on the optimal path.

We conclude that when condition (9) is violated and the project is feasible it must be the case that the optimal policy is to liquidate at some t when $\theta = 1$, and never to liquidate when $\theta = 0$.

Remark. If $S(\theta, K) = S + qK$ (i.e. the liquidation value does not depend on θ), the first best policy must be that liquidation never occurs. From Lemma 2, liquidation does not occur at $\theta = 0$, and if the liquidation value does not depend on θ, it cannot occur at $\theta = 1$ either. Thus, if the liquidation value does not depend on θ then the only projects with positive NPV are the ones for which (9) is satisfied.

Lemma 2 implies that the optimal policy is either never to liquidate or to liquidate only when $\theta = 1$. Consider the interval $\left[0, \frac{K^*}{d}\right]$ of capital values (it can never be optimal to have more capital than that). In this interval, the function $\widehat{W}(\theta, K)$ is linear in K,

[3] The strict inequality comes from the fact that capital must be strictly positive under the optimal policy at time 0, so the firm can sell $dK - \varepsilon$ and save strictly positive investment cost.

with slope d, while the function $S(1,K)$ is linear with slope $\overline{q}d$, which by assumption is smaller. Thus, if (9) is violated there is a value K^+ such that

$$\widehat{W}(1,0) + dK^+ = \overline{S} + \overline{q}dK^+,$$

i.e.

$$K^+ = \frac{\overline{S} - \widehat{W}(1,0)}{d(1-\overline{q})}.$$

Finally, define K^{**} as the value of capital that solves the equation

$$pR'(K^{**}) = 1 - \delta d(p\overline{q} + (1-p)). \tag{11}$$

We summarize the results in the next proposition.

Proposition 1 *When there are no agency problems the optimal investment policy can be characterized as follows.*

1. *If condition (9) is satisfied then the firm chooses at period 1 the level K^* defined in (6) and never liquidates. In subsequent periods the investment replaces the depreciated capital, i.e. $K_t - K_{t-1} = (1-d)K^*$.*
2. *If condition (9) is violated and $K^+ \geq K^{**}$ then the firm liquidates whenever $\theta_{t-1} = 1$, and otherwise chooses investment to reach the level $K^{**} < K^*$ defined in (11). Thus, $K_1 = K^{**}$ if $\theta_0 = 0$, and in subsequent periods $K_t - K_{t-1} = (1-d)K^{**}$, when $\theta_{t-1} = 0$.*
3. *If condition (9) is violated and $K^+ < K^{**}$ then the firm is liquidated at time 1 if $\theta_0 = 1$. If $\theta_0 = 0$ then the firm invests K^*, keeps the level of capital constant and never liquidates.*

It may appear strange that the firm is liquidated only after a favorable shock, while it is continued after an unfavorable shock. However, if S is interpreted as the resale value of the project this appears to be quite intuitive. Essentially, the project is started by an entrepreneur and, if successful, it is sold to a bigger firm which is better able to profit from it. On the other hand, if the project is only moderately successful, the best choice for the entrepreneur is to keep pursuing it until it becomes successful.

For future reference, we compute the value of the firm when liquidation at $\theta = 1$ is optimal. We have

$$W(0,0) = pR(K^{**}) - K^{**} + \delta\left(p\left(\overline{S} + \overline{q}dK^{**}\right) + (1-p)W(0,K^{**})\right)$$

and

$$W(0,K^{**}) = pR(K^{**}) - (1-d)K^{**} + \delta\left(p\left(\overline{S} + \overline{q}dK^{**}\right) + (1-p)W(0,K^{**})\right).$$

Solving for $W(0, K^{**})$ we get

$$W(0, K^{**}) = \frac{pR(K^{**}) - (1-d) K^{**}}{1 - \delta(1-p)} + \frac{\delta p \left(\overline{S} + \overline{q} d K^{**}\right)}{1 - \delta(1-p)}, \tag{12}$$

so that

$$W(0, 0) = \frac{pR(K^{**}) - K^{**} + \delta\left(p\left(\overline{S} + \overline{q} d K^{**}\right) + (1-p) d K^{**}\right)}{1 - \delta(1-p)}, \tag{13}$$

We now move to the analysis of the optimal policy under incomplete information.

3 Efficient Contracts under Incomplete Information

We start reminding the reader of the timing under incomplete information. At the beginning of period t a decision of liquidation or continuation is taken. The choice is based on the values of previous history h_{t-1}, including in particular the last report $\widehat{\theta}_{t-1}$. We denote by ℓ_t this decision, and $\Pr(\ell_t = L) = \alpha_t(h_{t-1})$. The value of α_t is dictated by the contract.

If the decision is to *liquidate*, the scrap value is $S_t = S(\theta_{t-1}, K_{t-1})$, the entrepreneur obtains $Q_t(h_{t-1}, S_t)$ and the lender $S_t - Q_t(h_{t-1}, S_t)$. If the decision is to *continue* then a new level of capital K_t is decided as the realization of the probability distribution $\kappa_t(h_{t-1})$. Both Q_t and the probability distribution κ_t are prescribed by the contract[4]. After capital is chosen, production takes place and the borrower observes $\theta_t R(K_t)$ and announces $\widehat{\theta}_t$. For simplicity, imagine that this happens at the middle of period t. The announcement will change the following elements:

1. The payment to the lender in period t, $\tau_t\left(\widehat{\theta}_t, K_t, h_{t-1}\right)$. In particular, $\tau_t(0, K_t, h_{t-1}) \leq 0$ because of limited liability.

2. The probability of liquidation at the beginning of period $t+1$, $\alpha_{t+1}(h_t)$ and the payment $Q_{t+1}(h_t, S_{t+1})$ in case of liquidation; notice that h_t includes now the report $\widehat{\theta}_t$ and the realization of $\kappa_t(h_{t-1})$; also, the value S_{t+1} is observed when the firm is liquidated and it depends on the actual θ_t.

3. The probability distribution on capital $\kappa_{t+1}(h_t)$ at the beginning of period $t+1$, if the firm is not liquidated.

We now introduce the following functions:

1. At the beginning of period t, history h_{t-1} is known and we denote $V(h_{t-1})$ the expected value for the entrepreneur.

[4]Using a probability distribution over K_t, rather than simply a value, is useful because it ensures the concavity of the value function. This will be made clearer in the discussion after Proposition 2.

2. At the beginning of period t a decision of liquidation or continuation is taken and, in case of continuation, a level of capital K_t is decided. We denote with $\widetilde{V}(K_t, h_{t-1})$ the expected value for the entrepreneur when the decision is to continue and the realization of $\kappa_t(h_{t-1})$ is K_t.

3. After K_t has been decided the entrepreneur observes θ_t and the level of production $\theta_t R(K_t)$ and decides a report $\widehat{\theta}_t$. We denote $\widehat{V}\left(\theta_t, \widehat{\theta}_t, K_t, h_{t-1}\right)$ the expected value for the entrepreneur at the middle of period t after history h_{t-1}, capital choice K_t, observation of θ_t and announcement of $\widehat{\theta}_t$.

Given our definitions, if the contract satisfies incentive compatibility so that the entrepreneur announces the true value of θ_t, we have

$$V(h_{t-1}) = \alpha_t Q_t + (1-\alpha_t) E_{\kappa(h_{t-1})}\left[\widetilde{V}(K_t, h_{t-1})\right],$$

where α_t is a function of h_{t-1} and Q_t is a function of h_{t-1} and S_t, and

$$\widetilde{V}(K_t, h_{t-1}) = p\widehat{V}(1, 1, K_t, h_{t-1}) + (1-p)\widehat{V}(0, 0, K_t, h_{t-1}). \tag{14}$$

The function $\widehat{V}\left(\theta_t, \widehat{\theta}_t, K_t, h_{t-1}\right)$ is given by

$$\widehat{V}\left(\theta_t, \widehat{\theta}_t, K_t, h_{t-1}\right) = \theta_t R(K_t) - \tau\left(\widehat{\theta}_t, K_t, h_{t-1}\right) + \delta V\left(\widehat{h}_t\right),$$

where $\widehat{h}_t = \left(h_{t-1}, \left(C, K_t, \widehat{\theta}_t\right)\right)$. The incentive compatibility constraint requires that

$$\widehat{V}(\theta_t, \theta_t, K_t, h_{t-1}) \geq \widehat{V}\left(\theta_t, \widehat{\theta}_t, K_t, h_{t-1}\right)$$

for each realization K_t and for each pair $\left(\theta_t, \widehat{\theta}_t\right)$. Limited liability implies $\tau(0, K_t, h_{t-1}) \leq 0$, so there must be histories h_{t-1} such that the entrepreneur pays a strictly positive amount when $\theta_t = 1$ (if not, the participation constraint of the lender would be violated). Thus, the incentive constraint is typically binding when $\theta_t = 1$. In order to convince the entrepreneur to report $\widehat{\theta}_t = 1$ when this is the true state of the world, the incentive compatibility constraint

$$\widehat{V}(1, 1, K_t, h_{t-1}) \geq R(K_t) + \delta V(h_{t-1}, (C, K_t, 0)) \tag{15}$$

has to be satisfied. Notice that we have assumed $\tau(0, K_t, h_{t-1}) = 0$, since negative values (equivalent to giving extra cash to the entrepreneur when $\theta = 0$ is announced) can never be optimal as they only worsen the incentive problem. Notice further that since

$$\widehat{V}(1, 1, K_t, h_{t-1}) = R(K_t) - \tau(1, K_t, h_{t-1}) + \delta V(h_{t-1}, (C, K_t, 1)),$$

condition (15) is equivalent to

$$\tau(1, K_t, h_{t-1}) \leq \delta\left[V(h_{t-1}, (C, K_t, 1)) - V(h_{t-1}, (C, K_t, 0))\right]. \tag{16}$$

We will use this form of the incentive compatibility constraint when we adopt a recursive representation of the optimal contract.

3.1 The Value of the Equity and the Value of the Firm

To give a recursive formulation to the problem, let $V = V(h)$ be the expected value promised to the entrepreneur at the beginning of period after history h. Since the lender has unlimited wealth, any non–negative amount of equity V can be attained simply by requiring that the project be liquidated and the borrower paid $Q = V$. Notice that this is true independently of the history h. On the other hand, negative values cannot be implemented due to the limited liability of the borrower. Thus, the set of values that V can take is $[0, +\infty)$. From now on, without loss of generality we will assume that the policy depends on the history h_{t-1} only via the promised equity value V and the accumulated level of capital K. Introducing additional variations based on history observed along the equilibrium path cannot increase the value of the firm (see Spear and Srivastava [15] for a justification of the recursive approach in dynamic moral hazard problems).

Denote now by $W(\theta, V, K)$ the value of the firm when the state of the world in the previous period is θ, a value V has to be given to the entrepreneur and the capital in the previous period is K.[5] Let K^n be the choice of capital in period t. Then the function $W(\theta, V, K)$ must satisfy the following functional equation

$$W(\theta, V, K) = \max_{\alpha,\kappa,\tau(\cdot),Q,V^H(\cdot),V^L(\cdot)} \alpha S(\theta, K) +$$

$$+ (1-\alpha) E_\kappa \left[pR(K^n) - I(\theta, K^n, K) + \delta \left(pW\left(1, V^H(K^n), K^n\right) + (1-p) W\left(0, V^L(K^n), K^n\right)\right)\right] \tag{17}$$

subject to

$$V = \alpha Q + (1-\alpha) E_\kappa \left[p\left(R(K^n) - \tau(K^n)\right) + \delta\left(pV^H(K^n) + (1-p) V^L(K^n)\right)\right] \tag{18}$$

$$\tau(K^n) \le \delta\left(V^H(K^n) - V^L(K^n)\right) \text{ each } K^n \in \text{supp}\kappa \tag{19}$$

$$\tau(K^n) \le R(K^n). \tag{20}$$

$$0 \le \alpha \le 1,\ Q \ge 0,\ V^H(K^n) \ge 0,\ V^L(K^n) \ge 0, \text{ each } K^n \in \text{supp}\kappa \tag{21}$$

where $V^L(K^n)$ and $V^H(K^n)$ are the new levels of equity promised to the entrepreneur when the new level of capital K^n is chosen and she announces $\theta = 0$ and $\theta = 1$ respectively, and $\tau(K^n)$ is the amount paid by the entrepreneur when announcing $\theta = 1$. Equality (18) is the promise-keeping constraint, inequality (19) is the form taken by the incentive compatibility constraint (16) under a recursive formulation and inequality (20) is the limited liability constraint. Finally, (21) collects the feasibility constraints.

The problem of finding the value function $W(\theta, V, K)$ can be decomposed in two parts. First, we can compute the value of the firm when continuation is imposed. Second, once that value has been obtained, we can use the continuation value and the liquidation value to compute the optimal liquidation policy.

[5] Notice that in the case of complete information the value of the firm does not depend on V; this is why in the previous section the value of the firm was given by a function $W(\theta, K)$.

Let $W_c(\theta, V_c, K)$ be the highest value of the firm that can be achieved when continuation is imposed, the state in previous period was θ, the level of capital is K and the entrepreneur is promised an equity value of V_c. Therefore $W_c(\theta, V_c, K)$ is obtained solving the problem

$$W_c(\theta, V_c, K) = \max_{\kappa, \tau(\cdot), V^H(\cdot), V^L(\cdot)} \quad E_\kappa \left[pR(K^n) - I(K^n, K, \theta)\right]$$

$$+\delta E_\kappa \left[pW\left(1, V^H(K^n), K^n\right) + (1-p) W\left(0, V^L(K^n), K^n\right)\right] \tag{22}$$

subject to

$$V_c = E_\kappa \left[p\left(R(K^n) - \tau(K^n)\right) + \delta\left(pV^H(K^n) + (1-p) V^L(K^n)\right)\right] \tag{23}$$

$$\tau(K^n) \leq \delta\left(V^H(K^n) - V^L(K^n)\right), \qquad \tau(K^n) \leq R(K^n) \tag{24}$$

$$V^H(K^n) \geq 0, \quad V^L(K^n) \geq 0 \text{ each } K^n \in \text{supp}\kappa \tag{25}$$

Notice that $W_c(\theta, V_c, K)$ is computed taking as given the function $W(\theta, V, K)$. Once we have the function W_c, we can rewrite the maximization problem as follows:

$$W(\theta, V, K) = \max_{\alpha, Q, V_c} \quad \alpha S(\theta, K) + (1-\alpha) W_c(\theta, V_c, K) \tag{26}$$

subject to

$$V = \alpha Q + (1-\alpha) V_c$$

$$Q \geq 0, \qquad 1 \geq \alpha \geq 0.$$

Standard results in dynamic programming imply that the solution to the functional equation (26) is unique (see Quadrini [12] for details). The remaining task is to characterize the properties of the functions W_c and W and of the optimal policy.

Inspecting problem (26) we can make a few simple observations. First, for each pair (V, K) we have $W(1, V, K) \geq W(0, V, K)$. This is because the constraint set is not affected by θ, while the objective function is non-decreasing in θ. (The inequality can be strict only if the optimal policy requires $\alpha > 0$ or $K^n < dK$ when $\theta = 1$). Second, it is obvious that whenever the optimal policy requires $\alpha(\theta, V, K) = 0$ then we have $W(\theta, V, K) = W_c(\theta, V, K)$. Third, if the optimal policy at a given V prescribes $K^n > dK$ for both values of θ then W_c is constant with respect to θ. Fourth, if $V = 0$ then the optimal policy prescribes $K^n = 0$ in each period in which the firm remains active with positive probability and $Q = 0$ in case of liquidation. In fact, the entrepreneur must be given at least $E_\kappa[pR(K^n)]$ in each period in which the firm is active. Thus, the only way to guarantee $V = 0$ is to set $K^n = 0$ with probability 1. Keeping the firm active with $K^n = 0$ can be optimal only if the expected value of the scrap value in the following period is higher than in the current period. This cannot happen if $\theta = 1$, thus implying that when $\theta = 1$ and $V = 0$ the optimal policy is to liquidate and the value of the firm is $W(1, 0, K) = S(1, K)$. If $\theta = 0$ then it is optimal to wait for liquidation until $\theta = 1$ if

$\underline{S} < \delta\left(p\overline{S} + (1-p)\,\underline{S}\right)$, or $\underline{S} < \frac{\delta p}{1-\delta(1-p)}\overline{S}$, and liquidate immediately otherwise. Thus, if we define

$$S^* = \max\left\{\underline{S}, \frac{\delta p}{1-\delta\,(1-p)}\overline{S}\right\} \tag{27}$$

then the value of the firm is $W(0,0,K) = S^* + q(\theta)\,dK$. Notice that this is strictly higher than $S(0,K)$ when $S^* > \underline{S}$.

The next proposition establishes further results about the functions W and W_c.

Proposition 2 *For each (θ, K) the functions $W(\theta, V, K)$ and $W_c(\theta, V_c, K)$ satisfy the following properties.*

1. *The functions $W(\theta, V, K)$ and $W_c(\theta, V, K)$ are non-decreasing in all arguments. For each (θ, K) the functions $W(\theta, \cdot, K)$ and $W_c(\theta, \cdot, K)$ are concave and the partial derivatives with respect to V are defined almost everywhere.*

2. *For each (θ, K) there is a value $V_{(\theta,K)}$ such that the function W is linear in V on the interval $\left[0, V_{(\theta,K)}\right]$. If $W_c(\theta, V, K)$ is constant with respect to θ then $V_{(1,K)} \geq V_{(0,K)}$.*

The proposition shows that the results of Clementi and Hopenhayn [2] and Quadrini [12] continue to hold (with some obvious modifications) when capital is durable and the liquidation value is stochastic: For given values of θ and K the value of the firm is non-decreasing and concave in the value of equity. The linear part of the value function corresponds to the case in which liquidation occurs with positive probability. The difference is that the threshold values for which liquidation occurs now depend on θ and K.

For a given value of K and θ, the set of values for V can be divided in three regions $[0, V_0)$, $[V_0, V_1]$ and $(V_1, +\infty)$ with the following characteristics.

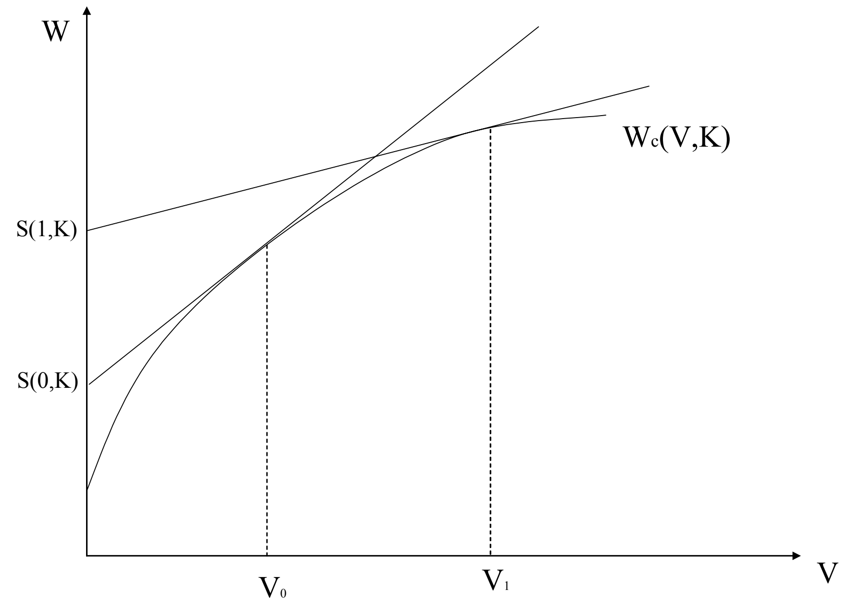
- When $V \in [0, V_0)$ then the firm is liquidated with probability $\underline{\alpha} = 1 - \frac{V}{V_0}$ when $\theta = 0$ and with probability $\overline{\alpha} = 1 - \frac{V}{V_1}$ when $\theta = 1$.

- When $V \in [V_0, V_1]$ then the firm is not liquidated when $\theta = 0$ and it is liquidated with probability $\overline{\alpha} = 1 - \frac{V}{V_1}$ when $\theta = 1$.

- When $V > V_1$ the firm continues with probability 1.

Notice that the values of V_0 and V_1 typically depend on θ and K. Notice also that sometimes for $\theta = 1$ it is optimal to liquidate with probability 1. In that case $V_1 = +\infty$.

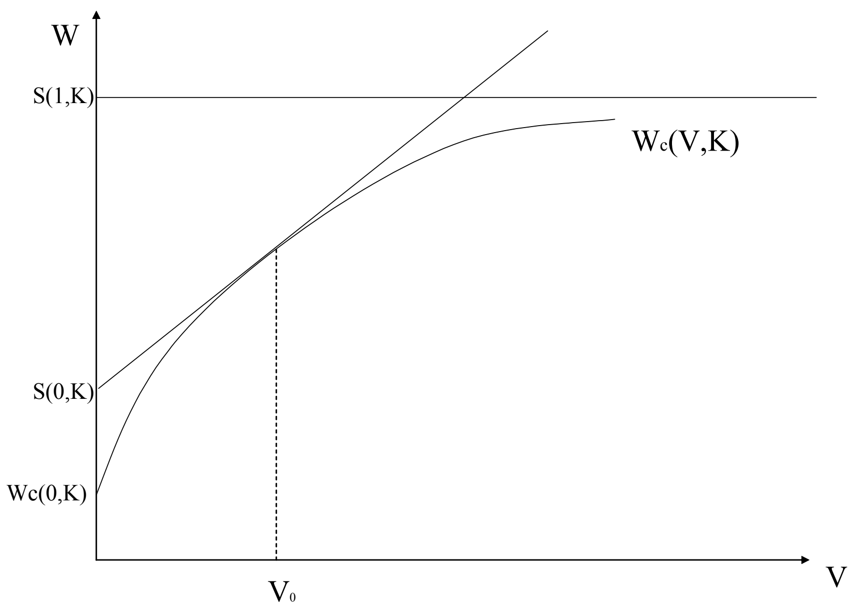
Figure 1 shows the case in which liquidation is undesirable (by this we mean that for a sufficiently high value of V liquidation does not occur), W_c does not depend on θ (no negative investment in case of continuation) and $S^* = \underline{S}$, where S^* is defined in (27). In this case the liquidation area is $[0, V_0]$ when $\theta = 0$ and $[0, V_1]$ when $\theta = 1$. Liquidation is more likely when $\theta = 1$ since in that case liquidation is less costly.



(1)

Liquidation areas when liquidation is undesirable.

Figure 2 shows the case in which liquidation occurs with probability 1 when $\theta = 1$. In this case immediate liquidation is preferable to continuation for every value V, i.e. $S(1, K) > W_c(1, V, K)$ for each value V. Thus, $V_{(1,K)} = +\infty$ and the value function is $W(1, V, K) = S(1, K)$.



(2)

Liquidation areas when liquidation is desirable.

In the rest of the paper we will focus on the impact that the introduction of durable capital and stochastic liquidation values has on the optimal policy of the firm.

4 Liquidation and Efficiency

When the liquidation value is not stochastic there is no liquidation under the first best policy. However, with a stochastic liquidation value, liquidation may occur under the first best policy when $\theta = 1$. It turns out that in such situations it may become possible to achieve the first best even in the presence of moral hazard. In fact, the first best can be achieved when the liquidation value under $\theta = 1$ is sufficiently large.

The intuitive explanation is that liquidation values are easier to monitor than output values, since we have assumed that the output $\theta R(K)$ is not verifiable, while the liquidation value $S(\theta, K)$ is. When the liquidation value is not stochastic this does not help to achieve the first best, since liquidation should never occur anyway. But with a stochastic liquidation value the first-best policy does sometime include liquidation and in those cases we can exploit the verifiability of liquidation values to lessen the incentive constraints.

4.1 Is the First Best Attainable?

The first best is implementable when it is possible to assign a value V to the entrepreneur at time zero such that the first best policy is implemented in every period and the optimal incentive compatible contract satisfies the individual rationality constraints for the lender and the entrepreneur.

We start from the following result. Remember that K^* is the first-best level of capital when liquidation is never optimal.

Proposition 3 *Suppose that liquidation does not occur under the first best. Then the first best policy is achievable if and only if* $V \geq \frac{pR(K^*)}{1-\delta}$.

One implication of the proposition is that the initial level of capital is not important in order to achieve the first best. Independently of the level of capital the first best can be achieved provided that the value of equity is high enough. Thus, if at any given moment in time the value V reaches $\frac{pR(K^*)}{1-\delta}$ then from that point on the optimal contract will implement the first best policy.

Another implication is that in this case the first best cannot be achieved at period 0, i.e. at least for some time the optimal second best policy must be different from the first best policy. Since the value of the project under the first best policy, given by (7), is

$$W^* = \delta \frac{pR(K^*) - (1 - d\delta) K^*}{1 - \delta} < \frac{pR(K^*)}{1 - \delta},$$

a contract assigning $V \geq \frac{pR(K^*)}{1-\delta}$ at time 0 would violate the individual rationality constraint for the lender. Notice however that the value $\frac{pR(K^*)}{1-\delta}$ can be reached after a sequence of positive shocks.

Suppose now that the first best policy requires investing K^{**}, obtained solving equation (11), whenever $\theta_{t-1} = 0$ and liquidating the firm at the beginning of period t whenever $\theta_{t-1} = 1$. Can the first best policy be implemented? The answer here is 'sometimes'. More specifically, let $W(0,0)$ be the value of the firm under the first best policy when $\theta = 0$, as defined in (13). Then the value of the project under the first best policy when the capital is zero is

$$W^* = \delta \left(p\overline{S} + (1-p) W(0,0) \right).$$

where $W(0,0)$ takes the value given by (13). We will show that when

$$W^* - A \geq \delta \frac{pR(K^{**})}{1-\delta} \tag{28}$$

the first best can be implemented.

Proposition 4 *Suppose that the first-best policy requires a level of capital K^{**} when $\theta = 0$ and liquidation when $\theta = 1$. Then the first best can be implemented if and only if condition (28) is satisfied*

To gain additional insights, use the value of $W(0,0)$ given in (13) to write condition (28) as

$$p\overline{S} - \frac{(1-\delta(1-p))}{\delta} A \geq \frac{p^2 R(K^{**})}{1-\delta} + (1-p)(1-\delta d((1-p)+p\overline{q})) K^{**}. \tag{29}$$

In general the condition is more easily satisfied when the liquidation value $\overline{S}$ is large and K^{**} is low. Thus, high liquidation values will make the implementation of the first best possible. The intuition is as follows. The announcement of $\theta_t = 1$ leads to liquidation in the following period; since the entrepreneur could announce $\theta_t = 0$ in each period and steal $R(K^{**})$ whenever possible, a mimimum value of $\frac{pR(K^{**})}{1-\delta}$ is necessary to pay the entrepreneur a sufficiently high value at liquidation to make the announcement incentive compatible. On the other hand the payment cannot be too high, since in that case the individual rationality constraint for the lender would be violated. The two goals are compatible when the liquidation value is high enough compared to what can be stolen in case of continuation, i.e. when $\overline{S}$ and $\overline{q}$ make the liquidation value sufficiently high or when a low level of K^{**} makes continuation not particularly attractive for the entrepreneur.

Notice that condition (28) applies in the extreme case in which the lender has to finance entirely the initial investment A. When the entrepreneur can contribute an initial capital M_B then the individual rationality condition for the lender is $W^* - M_L \geq \delta \frac{pR(K^{**})}{1-\delta}$, which is easier to satisfy since $M_L < A$.

4.2 The Second Best Policy and Efficient Liquidation

Even when the first-best is not attainable, the possibility of efficient liquidation still helps in getting the second best policy closer to the first best policy. Specifically, it turns out that liquidation with probability 1 occurs only when it is efficient.

Proposition 5 *For each level K, the optimal policy for the incomplete information case prescribes $\alpha = 1$ if and only if this is also the optimal policy for the complete information case.*

The idea is that, for any given level of capital, the value of continuation decreases when there are agency problems, but the value of liquidation does not. Thus, if liquidation is better than continuation under the first best policy, it must be better under the second best policy as well. On the other hand, if liquidation is implemented with probability 1 under the second best policy then it must be the case that liquidation is better than continuation for every possible value continuation value V_c. Otherwise, it would be better to randomize between liquidation and continuation with a value V_c. However, when V_c is sufficiently high the first best policy becomes implementable, so the value of continuation under incomplete information is identical to the value under complete information. Thus, liquidation must be the first best policy as well.

There are a couple of important caveats to Proposition 5. First, the proposition only states that liquidation with probability 1 will occur whenever it is efficient *for a given level of K*. It does not guarantee that the first-best level of K will actually be reached under the optimal second-best investment policy. Thus, even if liquidation with probability one occurs under the first and second best for the same sequence of shocks, the actual value of liquidation may be different. Second, under the first best policy liquidation either does not occur or it occurs with probability 1. Under the second best policy liquidation may occur with probability strictly positive but strictly less than one. In all such cases the optimal first-best policy would be not to liquidate.

5 The Effect of Size

We now analyze how the value function and the optimal policy vary with the size of the firm. The empirical results on the effect of size are summarized by Cooley and Quadrini [3] as follows:

> Conditional on age, the dynamics of firms (growth, volatility of growth, job creation, job destruction, and exit) are negatively related to the size of the firm.

Our model generates theoretical predictions which are broadly in agreement with these stylized facts. While age does not appear as variable in the objective function, age is positively correlated with V since firms with a low V have a higher probability of liquidation. Thus, information on the dynamics of the firm conditional on age can be obtained by fixing V and looking at how the value function and the optimal policy change with K.

We find that an increase in K, keeping V constant, decreases the probability of liquidation, since increasing size increases the continuation value more than the liquidation value. A bigger current size is positively correlated to bigger future size, since the cost of investment is reduced, but whenever the firm makes positive investment, the amount of the investment is negatively correlated to size. This follows from the fact that, conditional on making positive investment, the optimal level of capital is not related to existing capital, so that a higher current level of capital simply means that less investment is necessary to achieve the optimal level. Thus, growth in general depends negatively on size. The volatility of growth also depends negatively on size.[6]

5.1 Size and the Optimal Policy

Our first result establishes that, for a given V, the probability of liquidation decreases when the existing stock of capital increases. More precisely, remember that $V_{(\theta,K)}$ is the threshold value such that liquidation with prositive probability occurs only when $V < V_{(\theta,K)}$; then we have the following result.

Proposition 6 *The value $V_{(\theta,K)}$ does not increase in K and it strictly decreases if the optimal policy prescribes strictly positive investment at $\left(\theta, V_{(\theta,K)}, K\right)$. The value $V_{(\theta,K)}$ strictly increases in θ. For each pair (θ, V) the probability of liquidation does not increase in K and it strictly decreases if $\alpha(\theta, V, K) \in (0,1)$ and $V_{(\theta,K)}$ strictly decreases in K.*

The proposition shows that the probability of survival, other things equal, increases with the size of the firm. This is different from the result in Clementi and Hopenhayn. In their case the capital stock is not a state variable, and it is decided in every period as part of the optimal policy. The positive correlation of size and the probability of survival exists only because both size and the probability of survival are correlated to the value of equity: firms with a high equity value are on average larger and have a higher probability of survival. Notice furthermore that the positive relation between size and equity holds only on average in the Clementi–Hopenhayn model; for some value the optimal investment decreases[7] in V.

In our model size is measured by the existing stock of capital at the beginning of the period. It therefore makes sense to look at the effect of exogenous changes in size. The intuition for the result in Proposition 6 is that an increase of capital increases the value of continuation more than the value of liquidation. This is easier to see when the optimal policy prescribes strictly positive investment with probability 1 under continuation. In that case an additional amount of capital ΔK increases the value of continuation by $d\Delta K$ (since it reduces the necessary investment by that amount), while the value of liquidation increases only by $q(\theta)\, d\Delta K$ (the value at which capital can be sold). In general, the increase in the value of continuation is higher as long as there is a strictly positive probability of positive investment.

The next proposition establishes that, other things equal, a larger firm is more likely to choose a larger level of capital. Before we state the result, some notation is necessary.

[6] Our model cannot make predictions on job creation and destruction, since capital is the only factor of production. However, the model could be extended assuming a fixed labor-capital ratio, as in Cooley and Quadrini [3], to obtain a negative relation between size and job creation and destruction.

[7] The same is true in our model: the optimal choice K^n is not necessarily monotonic in V.

For a given state $s = (\theta, V, K)$ let $\widetilde{K}_s^n$ be the random variable chosen as optimal capital policy at s. Define Ξ_s as the support of $\widetilde{K}_s^n$, and let $\overline{K}_s^n$ and $\underline{K}_s^n$ be the supremum and the infimum of Ξ_s. Finally, for $\Delta > 0$ define the state $s + \Delta = (\theta, V, K + \Delta)$, i.e. the state with the same θ and V as s but a level of capital increased to $K + \Delta$, and let $\widetilde{K}_{s+\Delta}^n$, $\Xi_{s+\Delta}$ and $\underline{K}_{s+\Delta}^n$ and $\overline{K}_{s+\Delta}^n$ be defined in the obvious way. The next result shows that an increase in current size, defined as existing capital stock K at the beginning of period t, causes an increase in future size, defined as the optimal capital stock K^n chosen in period t. However a higher level of K in general requires less investment, since the optimal new level of capital K^n is closer.

Proposition 7 *Consider two states s and $s + \Delta$. Then*

1. *$\widetilde{K}_{s+\Delta}^n$ (weakly) first order stochastically dominate $\widetilde{K}_s^n$.*
2. *If $\underline{K}_s^n > d(K + \Delta)$ or $\underline{K}_s^n < dK$ then $\widetilde{K}_{s+\Delta}^n = \widetilde{K}_s^n$.*
3. *If $\underline{K}_s^n > d(K + \Delta)$ then the amount of investment decreases.*

The intuition for the first point is relatively simple. The cost of investment $I(\theta, K^n, K)$ is strictly decreasing in K. Thus, other things equal, it cannot be an optimal policy to choose a higher future level of capital stock when the current capital stock is lower.

The second point simply follows from the fact that increasing the level of capital when the investment is strictly positive or strictly negative does not change the constraint set and it is equivalent to adding a constant to the objective function. The optimal solution must therefore remain the same. The third point follows immediately. If the new capital level remains the same when the current capital stock increases, then the investment necessary to achieve the new capital level must decrease.

Notice that the proposition focuses exclusively on the role of past size in determining future size. It says that, other things equal, the future size of the firm does not decrease when the current size goes up. It does not say that current size is the only determinant of future size. It remains true that the optimal size typically depends also on V and θ. While, as in Clementi-Hopenhayn, the relation between K^n and V is not monotonic, we can say something about the effect of θ.

Proposition 8 *Consider the two states $s_0 = (0, V, K)$ and $s_1 = (1, V, K)$. Then $\widetilde{K}_{s_0}^n$ (weakly) first order stochastically dominate $\widetilde{K}_{s_1}^n$. If $\underline{K}_s^n > dK$ then $\widetilde{K}_{s_0}^n = \widetilde{K}_{s_1}^n$.*

The intuition here is that when $\theta = 1$ selling capital is more convenient. When the optimal policy at $\theta = 1$ involves only positive investment then this is irrelevant, so that the optimal policy remains the same at $\theta = 0$. Otherwise, the optimal plan requires selling more capital when $\theta = 1$ than when $\theta = 0$.

5.2 The Financial Consequences of Size

One implication of the previous results is that the expected return on assets of the firm is lower for bigger firms, and (under certain conditions) is also less volatile. To see this, define the return on assets for the firm when the current state is $s = (\theta, V, K)$ as

$$\widetilde{r}_t^s = \frac{\widetilde{Y}_t + \widetilde{W}_{t+1} - W_t}{W_t}, \tag{30}$$

where $\widetilde{Y}_t$ is the random variable determining the net income produced by the asset in the current period, $\widetilde{W}_{t+1}$ is the random variable determining the value of the firm at the beginning of next period and $W_t = E\left[\widetilde{Y}_t + \delta \widetilde{W}_{t+1}\right]$ is the value of the firm at the beginning of the current period. Let $\overline{r}_t^s$ be the mean of $\widetilde{r}_t^s$. In general the probability distribution of $\widetilde{r}_t^s$ depends on the optimal policy adopted, and it therefore changes with s.

Our first result shows that the expected return decreases with size.

Proposition 9 *Consider a state $s = (\theta, V, K)$, and let $\widetilde{r}^s$ be the random variable defined by (30). Then at each state $s+\Delta = (\theta, V, K + \Delta)$ with Δ small enough we have $E\left[\widetilde{r}^{s+\Delta}\right] \leq E\left[\widetilde{r}^s\right]$, with strict inequality if the optimal policy at s is such that $\underline{K}_s^n > dK$ or $\overline{K}_s^n < dK$.*

Notice that the predictions are made for a fixed value of V, i.e. we consider what happens to the firm when K is increased while V remains constant. This can be done only in the presence of durable capital.

To grasp the main intuition for the result, suppose that the optimal policy is single valued and investment is positive, i.e. there is a single optimal value K^n and $K^n > dK$, and there is no liquidation. If (θ, V, K) is the state at the end of time $t-1$, so that $W_t = W(\theta, V, K)$, then $\widetilde{Y}_t$ is the random variable

$$\widetilde{Y}_t = \begin{cases} R(K^n) - (K^n - dK) & \text{with prob. } p \\ -(K^n - dK) & \text{with prob. } 1-p, \end{cases}$$

while $\widetilde{W}_{t+1}$ is the random variable

$$\widetilde{W}_{t+1} = \begin{cases} W\left(1, V^H, K^n\right) & \text{with prob. } p \\ W\left(0, V^L, K^n\right) & \text{with prob. } 1-p \end{cases}$$

Then $W = E\left(\widetilde{Y}_t\right) + \delta E\left(\widetilde{W}_{t+1}\right)$, and the mean rate of return is

$$\overline{r} = \frac{E\left(\widetilde{Y}_t\right) + E\left(\widetilde{W}_{t+1}\right) - W}{W} = \frac{(1-\delta) E\left(\widetilde{W}_{t+1}\right)}{W}.$$

If we increase the amount of capital by ΔK the optimal policy does not change, as long as $K^n > d(K + \Delta K)$. The current value increases by $d\Delta K$, while the random variable

$\widetilde{W}_{t+1}$ is unaffected, since the optimal policy remains the same. Therefore, the new mean return is

$$\overline{r}' = \frac{(1-\delta)\, E\left(\widetilde{W}_{t+1}\right)}{W + d\Delta K} < \overline{r}.$$

Essentially, the point is that a larger firm has the same future values as a smaller form, since the optimal policy does not change. However the current value is higher, because lower investment is required, and this reduces the mean return.

A similar argument can be made for the volatility of returns.

Proposition 10 *Suppose the optimal policy at* $s = (\theta, V, K)$ *is such that* $\underline{K}_s^n > dK$ *or* $\overline{K}_s^n < dK$. *Then at each state* $s + \Delta = (\theta, V, K + \Delta)$ *with* Δ *small enough we have* $Var\left[\widetilde{r}^{s+\Delta}\right] < Var\left[\widetilde{r}^{s}\right]$.

The intuition is similar to the one discussed above for the average return. When the optimal future policy is fixed the future value are also fixed, and so is its variability. On the other hand, higher values of K reduce the need for investment and therefore increase the current value of the firm. The volatility of the return is basically obtained dividing the volatility of the future values by the current value; the numerator is fixed while the denominator increases when K increases, thus reducing the volatility of returns.

5.3 Capital Structure and Sensitivity of Investment to Cash Flow

We conclude our analysis with a couple of remarks on the sensitivity of capital structure to size and investment to cash flow. While we don't have formal results, the model can be used to provide some intuition about these issues.

The financial contracts considered in this paper are quite complicated. However we can interpret V as equity, and $B(\theta, V, K) = W(\theta, V, K) - V$ as debt[8].The evolution of the capital structure is complex, since in general K and V move together and in non-trivial ways along the optimal path. However, stretching somehow the model, assume that exogenous variations of K at any given moment are possible, e.g. because of exogenous changes in the prices of capital assets. In that case the value of V remains constant, so that the conclusion is that firms with higher level of capital have a capital structure more tilted towards debt. Furthermore, if we consider the case of non-stochastic policies and use the distance $V^H - V^L$ as a measure of the volatility of equity then higher levels K lead to higher value of K^n and, since $R(K^n) = \delta\left(V^H - V^L\right)$, to higher volatility. Thus, in general, bigger firms are more likely to be debt-financed and have more volatile equity. The conclusion that higher debt leads to a more volatile return on equity is of course familiar from the standard Modigliani-Miller theorem, but here it is reached for quite different reasons (remember that the Modigliani-Miller theorem does not hold in our model).

[8] This interpretation follows Clementi and Hopenhayn [2]. Other interpretations are possible; for example, the firm might be entirely financed with equity, with V being the value of the 'insider equity' held by the manager and B the value of 'external equity' held by outside financiers.

Consider now the sensitivity of investment to cash flow, a question that has received much attention in the literature. Under some conditions (see Hayashi [7]) the investment in any given period should be explained only by the Tobin's q, i.e. the ratio $\frac{W(\theta,V,K)}{A+K}$. Quadrini [12] points out that, in models with non-durable capital and random variables $\widetilde{\theta}_t$ which are independent across periods, Tobin's q is in fact a sufficient variable to explain investment. The reason is that the optimal investment is entirely explained by the value of equity V, which in turn determines the value of the firm $W(V)$. Thus, a regression of investment over Tobin's q and cash-flow should give a non-significant coefficient for cash flow. A positive coefficient for cash flow reappears when shocks are not independent; in that case the past value of shocks determines the future profitability of investment, so the optimal level of investment depends not only on V but also on θ_{t-1}. Since θ_{t-1} also determines the cash flow, the sensitivity of investment to cash-flow is reintroduced. Introducing durable capital is another way in which the sensitivity of investment to cash flow may reappear. With durable capital the state of the firm is defined by two variables, V and K. The optimal investment policy depends separately on both variables, not just on the way in which they influence the total value of the firm $W(\theta, V, K)$. Thus, a regression of investment over Tobin's q and cash-flow should typically give a non-zero coefficient for cash flow.

6 Conclusions

This paper has introduced stochastic liquidation values and durable capital in a model of optimal dynamic financing with moral hazard. A stochastic liquidation value makes it possible to have a positive probability of liquidation as part of the first-best policy, i.e. the value maximizing policy when there is no moral hazard. Furthermore, we show that under certain conditions it makes also possible to achieve the first best even with moral hazard.

Introducing durable capital allows us to make predictions on the impact of the capital stock over the optimal policy of the firm. Size has always been recognized as an important determinant of firm's behavior, but in absence of durable capital it is difficult to introduce size as a state variable. We show that, conditional on age, larger firms have a higher probability of survival, are more likely to be big in the future, have lower investment rates and lower average return and volatility on the assets. The results seem to be qualitatively consistent with the empirical literature.

Further research should extend the analysis in two directions. First, it would be perform a quantitative exercise, considering a parametric version of the model with realistic values for the parameters and checking whether the dynamics predicted by the model are quantitatively consistent with what has been found in empirical studies. Second, it would be interesting to move the analysis to the industry level, analyzing the endogenous determination of liquidation values as well as the impact of entry and exit.

Appendix

Proof of Lemma 1. We first show that it cannot be the case that at some time $t+1$ we have $K_{t+1} < dK_t$ for each value of θ_t. If that is the case then

$$E\left[\left.\frac{\partial I_{t+1}\left(\widetilde{\theta}_t, K_{t+1}, K_t\right)}{\partial K_t}\right| h_{t-1}\right] = -p\overline{q}d - (1-p)\,\underline{q}d. \tag{31}$$

Evaluating (5) at time t and time $t+1$ we have

$$pR'(K_t) = \frac{\partial I_t}{\partial K_t} + \delta E\left[\left.\frac{\partial I_{t+1}\left(\widetilde{\theta}_t, K_{t+1}, K_t\right)}{\partial K_t}\right| h_{t-1}\right].$$

$$pR'(K_{t+1}) = q(\theta_t) + \delta E\left[\left.\frac{\partial I_{t+2}\left(\widetilde{\theta}_{t+1}, K_{t+2}, K_{t+1}\right)}{\partial K_{t+1}}\right| h_t\right].$$

Since at time $t+1$ there is a sale, we have $K_{t+1} < dK_t$, hence by concavity $R'(K_{t+1}) > R'(K_t)$. Thus

$$\frac{\partial I_t}{\partial K_t} + \delta E\left[\left.\frac{\partial I_{t+1}\left(\widetilde{\theta}_t, K_{t+1}, K_t\right)}{\partial K_t}\right| h_{t-1}\right] < q(\theta_t) + \delta E\left[\left.\frac{\partial I_{t+2}\left(\widetilde{\theta}_{t+1}, K_{t+2}, K_{t+1}\right)}{\partial K_{t+1}}\right| h_t\right]$$

or

$$\frac{\partial I_t}{\partial K_t} - q(\theta_t) < \delta\left(E\left[\left.\frac{\partial I_{t+2}\left(\widetilde{\theta}_{t+1}, K_{t+2}, K_{t+1}\right)}{\partial K_{t+1}}\right| h_t\right] - E\left[\left.\frac{\partial I_{t+1}\left(\widetilde{\theta}_t, K_{t+1}, K_t\right)}{\partial K_t}\right| h_{t-1}\right]\right).$$

Consider the case $\theta_{t-1} = \theta_t$. The left hand side is either 0 (if there is negative investment) or $1 - q(\theta_t)$ (if there is positive investment). On the right hand side, the value of $E\left[\left.\frac{\partial I_{t+1}\left(\widetilde{\theta}_t, K_{t+1}, K_t\right)}{\partial K_t}\right| h_{t-1}\right]$ is given by (31). The value of $E\left[\left.\frac{\partial I_{t+2}\left(\widetilde{\theta}_{t+1}, K_{t+2}, K_{t+1}\right)}{\partial K_{t+1}}\right| h_t\right]$ will be $-d$ (if positive investment always occurs), $-pd - (1-p)\,\underline{q}d$ (if investment is positive at $\theta = 1$ and negative at $\theta = 0$), $-p\overline{q}d - (1-p)\,d$ (if investment is negative at $\theta = 1$ and positive at $\theta = 0$) or $-p\overline{q}d - (1-p)\,\underline{q}d$ (if investment is always negative). In all cases the right hand side is nonpositive, thus is cannot be strictly higher than the left hand side. We therefore have a contradiction.

Next, suppose that at a given time t it is optimal to have $K_t < dK_{t-1}$ when $\theta_{t-1} = 0$ but $K_t > dK_{t-1}$ when $\theta_{t-1} = 1$. This is impossible: if it is optimal to increase the amount of capital rather than selling when the price is high (i.e. $\theta_{t-1} = 1$), then it must be optimal to increase the amount of capital when the price is low (i.e. $\theta_{t-1} = 0$). Thus, the only possibility when sale occurs with positive probability at any time t is that there is a sale when $\theta_{t-1} = 1$ and no sale when $\theta_{t-1} = 0$.

Finally, suppose that at $\theta_{t-1} = 0$ we have $K_t > dK_{t-1}$ and at $\theta_t = 1$ we have $K_{t+1} < dK_t$. Then

$$pR'(K_t) = 1 - d\delta E[p\overline{q} + (1-p)]$$

and

$$pR'(K_{t+1}) = \overline{q} + \delta E\left[\left.\frac{\partial I_{t+2}\left(\widetilde{\theta}_{t+1}, K_{t+2}, K_{t+1}\right)}{\partial K_{t+1}}\right| h_{t-1}\right].$$

Again, by concavity $R'(K_{t+1}) > R'(K_t)$, therefore

$$\overline{q} + \delta E\left[\left.\frac{\partial I_{t+2}\left(\widetilde{\theta}_{t+1}, K_{t+2}, K_{t+1}\right)}{\partial K_{t+1}}\right| h_{t-1}\right] > 1 - d\delta E[p\overline{q} + (1-p)]. \tag{32}$$

We have previously shown that there is no period at which there is negative investment with probability 1. Thus, at $t+2$ there are two possibilities:

- investment is negative when $\theta_{t+1} = 1$ and positive investment when $\theta_{t+1} = 0$. In this case we have

  $$E\left[\left.\frac{\partial I_{t+2}\left(\widetilde{\theta}_{t+1}, K_{t+2}, K_{t+1}\right)}{\partial K_{t+1}}\right| h_{t-1}\right] = -d[p\overline{q} + (1-p)],$$

  so that inequality (32) is equivalent to $0 > 1 - \overline{q}$, which cannot be satisfied.

- investment is always positive. In this case we have

  $$E\left[\left.\frac{\partial I_{t+2}\left(\widetilde{\theta}_{t+1}, K_{t+2}, K_{t+1}\right)}{\partial K_{t+1}}\right| h_{t-1}\right] = -d,$$

  and inequality (32) becomes equivalent to

  $$d\delta[p\overline{q} + (1-p) - 1] > 1 - \overline{q}.$$

  This is impossible since the LHS is non-positive and the RHS is strictly positive.

This conclude the proof. ■

Proof of Lemma 2. Let t^* be the first time at which liquidation occurs at $\theta = 0$, and let K_{t^*-1} be the amount of capital at the beginning of date t^*. It must be

$$S(0, K_{t^*-1}) \geq W_c(0, K_{t^*-1}),$$

At time 1, right after the firm has been established, liquidation at $\theta = 0$ cannot be optimal, since otherwise (by Assumption 3) the firm would not be profitable. This implies

$$W_c(0,0) > \underline{S} \tag{33}$$

and an optimal level $K_1(0) > 0$ of investment. The value $W_c(0,0)$ is obtained under the optimal investment and liquidation plan.

Consider now the problem at the beginning of time t^*. The firm has capital dK_{t^*-1}. The firm can always adopt the same plan as at time zero, except that it will invest $K_1 - dK_{t^*-1}$ instead of K_1, or sell the quantity $dK_t^* - K_1$ if it is positive. When $K_1 \geq dK_{t^*-1}$ then the firm gets at least $W_c(0,0) + dK_{t^*-1}$, since it uses the same policy as at time zero but saves dK_{t^*-1} in investment. If $K_1 < dK_{t^*-1}$ then the firm saves K_1 and also gets $\underline{q}\left(dK_{t-1}^* - K_1\right)$, so that the value is at least $W_c(0,0) + \left(1-\underline{q}\right)K_1 + \underline{q}dK_{t^*-1}$. Since liquidation is optimal we have either

$$\underline{S} + \underline{q}dK_{t^*-1} \geq W_c(0,0) + dK_{t^*-1}. \tag{34}$$

or

$$\underline{S} + \underline{q}dK_{t^*-1} \geq W_c(0,0) + \left(1-\underline{q}\right)K_1 + \underline{q}dK_{t^*-1}. \tag{35}$$

Inequalities (33) and (34) are compatible only if $\underline{q} > 1$, but we assumed that the opposite is true. Inequalities (33) and (35) are compatible only $\left(1-\underline{q}\right)K_1 < 0$, which is unfeasible. ■

Proof of Proposition 1. The first point has already been proven. We now prove the remaining 2.

If condition (9) is violated then we have $\widehat{W}(1,0) < \overline{S}$. Thus, when $\theta_0 = 1$ the project is liquidated. The problem at time 1 when $\theta_0 = 0$ has been observed and we assume that the firm is always liquidated at $\theta = 1$ is therefore

$$W(0,0) = \max_{K \geq 0} \quad pR(K) - K + \delta\left[p\left(\overline{S} + \overline{q}dK\right) + (1-p)W(0,K)\right].$$

Since $\delta p\overline{S}$ is constant, this is equivalent to solving the problem

$$W(0,0) = \max_{K \geq 0} \quad pR(K) - (1-\delta p\overline{q}d)K + \delta(1-p)W(0,K).$$

Since, by Lemma 2, the firm is never liquidated when $\theta_{t-1} = 0$, the problem becomes equivalent to solving one in which liquidation never occurs, the discount rate is $\delta(1-p)$ rather than δ and the price of capital is $(1-\delta p\overline{q}d)$ rather than 1. Thus, the optimal policy will be to reach immediately the value K^{**} such that

$$pR'(K^{**}) = 1 - \delta d(p\overline{q} + (1-p)).$$

At time t, the firm replaces the depreciated capital (i.e., $I_t = (1-d)K^{**}$) whenever $\theta_{t-1} = 0$, and liquidates the firm at $\overline{S} + \overline{q}K^{**}$ otherwise.

Notice that if $K^{+} \geq K^{**}$ then it is actually optimal to liquidate when $\theta_{t-1} = 1$ and $K_{t-1} = K^{**}$, thus confirming the optimality of the liquidation policy. If $K^{+} < K^{**}$ then liquidation cannot be optimal when $\theta = 1$ and capital is K^{**}. Since $K^{*} > K^{**}$ and for levels of capital $K \leq \frac{K^{*}}{d}$ both the value of continuation and liquidation increase linearly, we conclude that the optimal policy must be to continue for each value of θ, so that the optimal level of capital is K^{*}. Notice however that since (9) is violated, liquidation when $\theta = 1$ is optimal when the level of capital is $K = 0$. Thus the optimal policy involves liquidation only if $\theta_0 = 1$, and a constant level of capital K^{*} otherwise. ■

Proof of Proposition 2. Since we will use results from Stokey, Lucas and Prescott [13], we first recast the problem using a similar notation, so that the way in which their results are applied is clearer. Define the vector of control variables as

$$x = \left(\alpha_x, Q_x, K_x, \tau_x, V_x(0), V_x(1)\right).$$

Given a choice $y = \left(\alpha_y, Q_y, K_y, \tau_y, V_y(0), V_y(1)\right)$, the return function is defined as

$$F(x, y, \theta) = \alpha_y S(\theta, K_x) + (1 - \alpha_y)\left[pR(K_y) - I(\theta, K_x, K_y)\right].$$

Define Γ as the set of vectors $\left(\alpha_y, Q_y, K_y, \tau_y, V_y(0), V_y(1)\right)$ that satisfy the following:

$$\alpha_y \in [0,1], \qquad Q_y \geq 0, \qquad K_y \geq 0$$

$$\tau_y \leq \min\left\{\delta\left(V_y(1) - V_y(0)\right), R(K_y)\right\}$$

$$V_y(0) \geq 0, \qquad V_y(1) \geq 0.$$

and $\Delta\Gamma(x, \theta)$ the set of probability distributions over Γ that satisfy

$$V_x(\theta) = E_\gamma\left[\alpha_y Q_y + (1 - \alpha_y)\left[p\left(R(K_y) - \tau_y\right) + \delta\left(pV_y(1) + (1-p)V_y(0)\right)\right]\right],$$

[*¿ES $V_x(\theta)$ O $V_y(\theta)$?*] where $\gamma \in \Delta\Gamma(x, \theta)$ denotes a probability distribution over Γ.

The value function can be written as

$$W(x, \theta) = \max_{\gamma \in \Delta\Gamma(x)} \quad (1 - \alpha_x) E_{\gamma, \theta'}\left[\left(F(x, y, \theta) + \delta E\left[W\left(y, \theta'\right)\right]\right)\right]$$

Standard results in dynamic programming imply that $W(x, \theta)$ exists and is unique. It is also clear that

$$W\left(\theta, \left(\alpha_x, Q_x, K_x, \tau_x, V_x(0), V_x(1)\right)\right)$$

can actually be written as

$$W^{*}\left(\theta, \left(\alpha_x, K_x, V_x(\theta)\right)\right)$$

since neither $\Delta\Gamma(x, \theta)$ nor F depend on Q_x, τ_x and $V_x\left(\theta'\right)$ when $\theta' \neq \theta$. Finally, we also have

$$W^{*}\left(\theta, \left(\alpha_x, K_x, V_x(\theta)\right)\right) = (1 - \alpha_x) W^{*}\left(\theta, \left(0, K_x, V_x(\theta)\right)\right).$$

Thus, define the function

$$W(\theta, V, K) = W^*(\theta, (0, K, V))$$

The function $W(\theta, V, K)$ is the one that we have been discussing in the text.

The function is increasing in K, because the return function is increasing in K_x and the constraint set $\Delta\Gamma(\theta, x)$ does not depend on K_x. To see that the function is increasing in V, notice that the return only depends on α and K, but not on Q or τ. When V is increased it remain possible to use the same policies for α and K, achieving the higher V through decreases in τ or increases in Q. Thus, increasing V expands the set of payoff-relevant policies. A similar argument establishes that W_c is increasing in V_c.

To see that $W(\theta, V, K)$ is concave in V when θ and K are fixed, suppose that there are two values V_1 and V_2 such that

$$qW(\theta, V_1, K) + (1-q)W(\theta, V_2, K) > W(\theta, qV_1 + (1-q)V_2, K) \tag{36}$$

for some $q \in (0,1)$. For the given q, consider the value $V_q = qV_1 + (1-q)V_2$. The value V_q can be promised to the entrepreneur by offering the policy implemented at V_1 with probability q and the policy implemented at V_2 with probability $(1-q)$; notice that such policies are clearly feasible. The expected value of the firm in that case would be the left hand side of (36). This is greater than the right hand side, contradicting the claim that $W(\theta, qV_1 + (1-q)V_2, K)$ is the highest value of the firm that can be achieved while giving V_q to the entrepreneur. A similar argument establishes the concavity of W_c. (Notice that the argument cannot be applied to establish the concavity with respect to K given (θ, V), since K enters directly the return function.) Since $W(\theta, V, K)$ and $W_c(\theta, V, K)$ are increasing and concave in V, the partial derivatives $\frac{\partial W}{\partial V}$ and $\frac{\partial W_c}{\partial V}$ are defined almost everywhere.

Finally, the proof of part (2) is the same as in Clementi and Hopenhayn, with the only change that now the upper bound of the region over which $W(\theta, \cdot, K)$ is linear depends on (θ, K). ■

Proof of Proposition 3. If the first best is implemented then an investment K^* must occur in every period independently of the history of announcements. This implies that the entrepreneur can achieve a value $\frac{pR(K^*)}{1-\delta}$ simply by announcing $\theta = 0$ in every period and stealing the output. Thus, in order to implement the first best policy we need $V \geq \frac{pR(K^*)}{1-\delta}$.

When this condition is satisfied the first best can be achieved by a policy of investing K^* in every period independently of past history, paying $V - \frac{pR(K^*)}{1-\delta}$ immediately to the entrepreneur, and giving the entire output $\theta_t R(K^*)$ to the entrepreneur in each period. ■

Proof of Proposition 4. The optimal policy requires to achieve a level of capital K^{**} whenever the announcement at $t-1$ has been 0, and to liquidate otherwise. One possible reporting policy for the entrepreneur is to announce $\theta_t = 0$ at each period. This gives an expected utility equal to

$$V^* = \delta \frac{pR(K^{**})}{1-\delta}.$$

Thus, any feasible contract that implements the first best must give at least V^* to the entrepreneur. The individual rationality constraint of the lender then requires

$$W^* - V^* \geq A,$$

thus confirming that the condition is necessary.

We now show that the condition is sufficient by describing a feasible policy that achieves the first best. Suppose that investment follows the first best policy. Payments are as follows:

- If $\theta_t = 0$ then the entrepreneur pays zero and the firm continues.
- If $\theta_t = 1$ then the entrepreneur pays zero. In the following period the firm is liquidated and the entrepreneur is paid $Q = \frac{pR(K^{**})}{1-\delta}$ if $S = S(1, K^{**})$ (i.e. the entrepreneur told the truth), and zero otherwise.

We will show that this payment policy is incentive compatible and gives the entrepreneur exactly V^*. Define

$$\widetilde{V}^L = pV^H + (1-p)\,V^L$$

$$V^H = R(K^{**}) + \delta Q \qquad\qquad V^L = \delta \widetilde{V}^L$$

so that

$$\widetilde{V}^L = p\left(R(K^{**}) + \delta Q\right) + (1-p)\,\delta\widetilde{V}^L \qquad \Longrightarrow \qquad \widetilde{V}^L = \frac{pR(K^{**})}{1-\delta(1-p)} + \frac{\delta p}{1-\delta(1-p)}Q.$$

Incentive compatibility is satisfied if

$$R(K^{**}) + \delta Q \geq R(K^{**}) + \delta\widetilde{V}^L \qquad \Longrightarrow \qquad Q \geq \widetilde{V}^L$$

which is satisfied given the definition of Q.

At time 0 capital is zero. Announcing $\theta = 1$ gives Q in the following period, while announcing $\theta = 0$ causes an investment of K^{**} and gives a utility of $\widetilde{V}^L$. Thus, the present expected value for the entrepreneur is

$$V = \delta\left(pQ + (1-p)\,\widetilde{V}^L\right) = \delta\frac{pR(K^{**})}{1-\delta}.$$

We conclude that the proposed policy achieves the first best, it's incentive compatible and individually rational for both the lender and the entrepreneur. ■

Proof of Proposition 5. Let $W_c^*(\theta, K)$ be the value function in case of continuation when there is complete information, and $W_c(\theta, V_c, K)$ the value function in case of incomplete information. Notice that $W_c(\theta, V_c, K) \leq W_c^*(\theta, K)$ for each V_c, since the first function is obtained solving a maximization problem with more constraints. Propositions

3 and 4 imply that it is possible to find V^+ such that $W_c(\theta, V_c, K) = W_c^*(\theta, K)$ for each K, θ and $V_c \geq V^+$.

Now observe that if $\alpha = 1$ is optimal in the complete information case it must be the case that $S(1, K) \geq W_c^*(1, K)$. Therefore $S(1, K) \geq W_c^*(1, V_c, K)$ and liquidation is optimal in the incomplete information problem. This proves the 'if' part.

To prove the 'only if' part, suppose that the optimal policy under incomplete information is $\alpha = 1$ and $W_c^*(1, K) > S(1, K)$. It must be the case that $S(1, K) \geq W_c(1, V_c, K)$ for each V_c, otherwise we can find $\alpha < 1$ and $V_c = \frac{V}{1-\alpha}$ such that $W_c(\theta, V_c, K) > S(1, K)$, that is, we reach a greater value of the firm if we don't liquidate, contradicting optimality of $\alpha = 1$. But this is not possible, since $W_c(\theta, V_c, K) = W_c^*(\theta, K)$ for $V_c \geq V^+$. ■

Proof of Proposition 6. The function $W_c(\theta, V, K)$ is almost everywhere differentiable and by the envelope theorem

$$\frac{\partial W_c(\theta, V, K)}{\partial K} = d \Pr(K^n \geq dK) + q(\theta) d \Pr(K^n < dK).$$

The liquidation value $S(\theta, K)$ is linear in K and

$$\frac{\partial S(\theta, K)}{\partial K} = q(\theta) d.$$

Thus $\frac{\partial S}{\partial K} \leq \frac{\partial W_c}{\partial K}$ for each V. Since $V_{(\theta,K)}$ is the point at which the line with intercept $S(\theta, K)$ is tangent to $W_c(\theta, V, K)$, if the function W_c increases no less than the value $S(\theta, K)$ then the point $V_{(\theta,K)}$ cannot increase. In particular, if at $\left(V_{(\theta,K)}, K\right)$ we have $\Pr(K^n > dK)$ then $\frac{\partial W_c\left(\theta, V_{(\theta,K)}, K\right)}{\partial K} > \frac{\partial S(\theta,K)}{\partial K}$, so that the value of $V_{(\theta,K)}$ decreases as K increases.

To see the effect of a change in θ, observe that when we move from $\theta = 0$ to $\theta = 1$ the liquidation value increases by $\overline{S} - \underline{S} + \left(\overline{q} - \underline{q}\right) dK$. Now let $W_c^{\#}(0, V, K)$ be the value of continuation when $\theta = 0$ but the policy prescribed for $(1, V, K)$ is adopted. Since such policy is not necessarily optimal we have $W_c^{\#}(0, V, K) \leq W_c(0, V, K)$. Therefore

$$W_c(1, V, K) - W_c(0, V, K) \leq W_c(1, V, K) - W_c^{\#}(0, V, K) \leq \left(\overline{q} - \underline{q}\right) dK.$$

We conclude that $S(1, K) - S(0, K) > W_c(1, V, K) - W_c(0, V, K)$. Since $V_{(\theta,K)}$ is the point at which the line with intercept $S(\theta, K)$ is tangent to $W_c(\theta, V, K)$ and S increases more than W_c when θ increases, we conclude that $V_{(1,K)} > V_{(0,K)}$.

The probability of liquidation can change with K only at points (θ, V, K) at which $\alpha(\theta, V, K) < 1$. At such points we have

$$\alpha(\theta, V, K) = 1 - \frac{V}{V_{(\theta,K)}}$$

and the conclusion therefore follows from the results on $V_{(\theta,K)}$. ■

Proof of Proposition 7. Consider problem (22). We want to apply Theorem 4 in Milgrom and Shannon [11], and we will do so by showing that the objective function is quasi-supermodular in the decision variables $\left(\kappa, \tau(\cdot), V^H(\cdot), V^L\right)$ and it satisfies increasing difference in $\left(\left(\kappa, \tau(\cdot), V^H(\cdot), V^L\right); K\right)$.

The space where $\left(\kappa, \tau(\cdot), V^H(\cdot), V^L\right)$ is defined is the Cartesian product of the space of probability distributions κ on $[0, +\infty)$, the space of functions $\tau(x)$ such that $\tau(x) \leq R(x)$ each x and the space of non-negative functions V^H and V^L. We define the ordering on this spaces as follows:

1. $\kappa \preceq \kappa'$ if κ' first order stochastically dominates κ.
2. $\tau \preceq \tau'$ if $\tau(x) \leq \tau'(x)$ each x, and similarly for V^H and V^L.
3. $\left(\kappa, \tau(\cdot), V^H(\cdot), V^L\right) \preceq \left(\kappa', \tau'(\cdot), V^{H\prime}(\cdot), V^{L\prime}\right)$ if each component of the first vector is lower than the corresponding component of the second vector.

Since the objective function does not depend on τ and it is increasing in both V^H and V^L (thus implying quasi-supermodularity) we only have to prove quasi-supermodularity with respect to κ. For convenience, we remind here the reader of some basic definitions needed to apply the Milgrom-Shannon theorem.

Given a partially ordered set X and two elements x,y in X, we define $x \wedge y$ as the largest element of X such that $x \wedge y \preceq x$ and $x \wedge y \preceq y$. Similarly, $x \vee y$ is the smallest element in X such that $x \preceq x \vee y$ and $y \preceq x \vee y$. The set X is a lattice if, given x, y in X, we have that $x \wedge y$ and $x \vee y$ are also in X. A function f defined on the lattice X is quasi-supermodular if, given two elements $x, y \in X$, whenever the inequality $f(x) \geq f(x \wedge y)$ is satisfied we also have $f(x \vee y) \geq f(y)$.

Let now consider the space of probability distribution on the positive real line endowed with the first-order stochastic dominance order. Consider two distribution κ and κ' represented by the cumulative distribution functions F and G respectively. Then $\kappa \vee \kappa'$ has cumulative distribution function $H(x) = \min\{F(x), G(x)\}$, while $\kappa \wedge \kappa'$ has cumulative distribution function $L(x) = \max\{F(x), G(x)\}$. We will prove that for any function $f(x)$, if $\int f(x)\, dF \geq \int f(x)\, dL$ then $\int f(x)\, dH \geq \int f(x)\, dG$. The first inequality can be written as

$$\int f(x)\, dF \geq \int_{\{x|F(x) \geq G(x)\}} f(x)\, dF + \int_{\{x|F(x)<G(x)\}} f(x)\, dG \tag{37}$$

or

$$\int_{\{x|F(x)<G(x)\}} f(x)\, dF \geq \int_{\{x|F(x)<G(x)\}} f(x)\, dG.$$

The second inequality can be written as

$$\int_{\{x|F(x) \geq G(x)\}} f(x)\, dG + \int_{\{x|F(x)<G(x)\}} f(x)\, dF \geq \int f(x)\, dG \tag{38}$$

or

$$\int_{\{x|F(x)<G(x)\}} f(x)\,dF \geq \int_{\{x|F(x)<G(x)\}} f(x)\,dG.$$

But this implies that whenever inequality (37) is satisfied, inequality (38) must also be satisfied. Thus any function defined on the real numbers is quasi-supermodular when defined over the space of probability distributions over the real line.

Finally, to prove that the objective function satisfies increasing difference in $(\kappa; K)$ we have to show that the difference between the objective function at K' and the objective function computed at K is increasing in κ whenever $K' > K$. To see this, observe that such difference is given by the function

$$I(\theta, K^n, K) - I\left(\theta, K^n, K'\right) = \begin{cases} d(K' - K) & \text{if} \quad K^n \geq dK' \\ (K^n - dK) - q(\theta)(K^n - dK') & \text{if} \quad dK' > K^n \geq dK \\ q(\theta)\, d(K' - K) & \text{if} \quad dK > K^n, \end{cases}$$

which is increasing in K^n. Therefore, $E_\kappa[I(\theta, K^n, K) - I(\theta, K^n, K')]$ is increasing in κ. This proves that the optimal policy κ is increasing in K.

Similarly, if we fix K, the difference between the objective function at $\theta = 1$ and at $\theta = 0$ is

$$I(0, K^n, K) - I(1, K^n, K) = \begin{cases} 0 & \text{if} \quad K^n \geq dK \\ \left(\underline{q} - \overline{q}\right)(K^n - dK) & \text{if} \quad dK > K^n, \end{cases}$$

so that it is decreasing in K^n. Therefore, $E_\kappa[I(0, K^n, K) - I(1, K^n, K)]$ is decreasing in κ. This proves that the optimal policy κ is decreasing in θ.

Suppose now that $\underline{K}_s^n > dK$, so that at s investment is always strictly positive. Suppose now that the quantity of capital is increased by a small amount Δ such that the inequality $\underline{K}_s^n > d(K + \Delta)$ still holds. It must be the case that the optimal policy remains the same, so that the value of the value function increases by $d\Delta$. If this were not the case then we would have $W(\theta, V, K + \Delta) > W(\theta, V, K) + d\Delta$, but this implies that by adopting at state (θ, V, K) the policy adopted $(\theta, V, K + \Delta)$ we would get a value strictly higher than $W(\theta, V, K)$, a contradiction. A similar reasoning implies that the optimal policy does not change when $\overline{K}_s^n < dK$ and we increase K by Δ.

At last, the third point follows immediately from the fact that the optimal policy $\widetilde{K}^n$ remains the same when K increases to $K + \Delta$. ■

Proof of Proposition 8. Using the framework of the proof of Proposition 7, the difference between the objective function at state $(1, V, K)$ and at state $(0, V, K)$ is

$$I(0, K^n, K) - I(1, K^n, K) = \begin{cases} 0 & \text{if} \quad K^n \geq dK \\ \left(\underline{q} - \overline{q}\right)(K^n - dK) & \text{if} \quad dK > K^n, \end{cases}$$

so that it is decreasing in K^n. Therefore, $E_\kappa[I(0, K^n, K) - I(1, K^n, K)]$ is decreasing in κ. This proves that the optimal policy κ is decreasing in θ.

If the optimal policy at $(1, V, K)$ involves only positive investment then the value of the objective function does not depend on θ. Since the value function is increasing in θ, the optimal policy must be the same at $(0, V, K)$. ■

Proof of Proposition 9. For a given state $s = (\theta, V, K)$ the variable $\widetilde{Y}$ is given by

$$\widetilde{Y} = \begin{cases} S(\theta, K) & \text{with prob. } \alpha \\ \widetilde{\theta} R\left(\widetilde{K}^n\right) - I\left(\widetilde{\theta}, \widetilde{K}^n, K\right) & \text{with prob. } 1-\alpha \end{cases}$$

and the variable $\widetilde{W}$ is given by

$$\widetilde{W} = \begin{cases} 0 & \text{with prob. } \alpha \\ W\left(\widetilde{\theta}, V\left(\widetilde{\theta}, \widetilde{K}^n\right), \widetilde{K}^n\right) & \text{with prob. } 1-\alpha. \end{cases}$$

Let $\widetilde{Y}_\Delta$, $\widetilde{W}_\Delta$, $\widetilde{K}^n_\Delta$, α_Δ be the variables corresponding to the state $s + \Delta = (\theta, V, K + \Delta)$ and let

$$\overline{Y} = \alpha S(\theta, K) + (1-\alpha) E\left[pR\left(\widetilde{K}^n\right) - I\left(\theta, \widetilde{K}^n, K\right)\right],$$

$$\overline{Y}_\Delta = \alpha_\Delta S(\theta, K+\Delta) + (1-\alpha_\Delta) E\left[pR\left(\widetilde{K}^n_\Delta\right) - I\left(\theta, \widetilde{K}^n_\Delta, K+\Delta\right)\right],$$

$$\overline{W} = (1-\alpha) E\left[W\left(\widetilde{\theta}, V\left(\widetilde{K}^n\right), \widetilde{K}^n\right)\right],$$

$$\overline{W}_\Delta = (1-\alpha_\Delta) E\left[W\left(\widetilde{\theta}, V\left(\widetilde{K}^n_\Delta\right), \widetilde{K}^n_\Delta\right)\right],$$

so that

$$W(s) = \overline{Y} + \delta\overline{W} \qquad\qquad W(s+\Delta) = \overline{Y}_\Delta + \delta\overline{W}_\Delta.$$

The expected returns are

$$\overline{r} = \frac{\overline{Y} + \overline{W} - W(s)}{W(s)} = \frac{(1-\delta)\overline{W}}{W(s)}$$

$$\overline{r}_\Delta = \frac{\overline{Y}_\Delta + \overline{W}_\Delta - W(s+\Delta)}{W(s+\Delta)} = \frac{(1-\delta)\overline{W}_\Delta}{W(s+\Delta)}.$$

Therefore, the condition $\overline{r} \geq \overline{r}_\Delta$ is equivalent to

$$\frac{W(s+\Delta)}{W(s)} \geq \frac{\overline{W}_\Delta}{\overline{W}}.$$

Suppose first that

$$\frac{\overline{Y}_\Delta}{\overline{Y}} \geq \frac{\overline{W}_\Delta}{\overline{W}}$$

This implies

$$\frac{\frac{\overline{Y}_\Delta}{\overline{W}_\Delta}}{\frac{\overline{Y}}{\overline{W}}} \geq 1 \rightarrow \frac{\frac{\overline{Y}_\Delta}{\overline{W}_\Delta} + \delta}{\frac{\overline{Y}}{\overline{W}} + \delta} \geq 1 \rightarrow \frac{\overline{Y}_\Delta + \delta\overline{W}_\Delta}{\overline{Y} + \delta\overline{W}} \geq \frac{\overline{W}_\Delta}{\overline{W}}$$

$$\rightarrow \frac{W(s+\Delta)}{W(s)} \geq \frac{\overline{W}_\Delta}{\overline{W}},$$

so that we are done. Next assume

$$\frac{\overline{Y}_\Delta}{\overline{Y}} < \frac{\overline{W}_\Delta}{\overline{W}}. \tag{39}$$

If this is the case we must have

$$\frac{\overline{Y}_\Delta}{\overline{Y}} < \frac{W(s+\Delta)}{W(s)} < \frac{\overline{W}_\Delta}{\overline{W}},$$

which in turn implies

$$\frac{\overline{Y}_\Delta}{W(s+\Delta)} < \frac{\overline{Y}}{W(s)}. \tag{40}$$

Let $\Delta W = W(s+\Delta) - W(s)$ and $\Delta Y = \overline{Y}_\Delta - \overline{Y}$. Then inequality (40) can be written as

$$\frac{\Delta Y}{\Delta W} < \frac{\overline{Y}}{W(s)}.$$

If $\alpha < 1$ then $\frac{\overline{Y}}{W(s)} < 1$. However, by the envelope theorem, we have

$$\lim_{\Delta K \to 0} \frac{\Delta Y}{\Delta W} = 1.$$

Thus, for ΔK sufficiently small inequality (39) can't be satisfied. ■

Proof of Proposition 10. Suppose first that at state s we have $\underline{K}_s^n > dK$, i.e. investment is always positive. In this case the variance of the return is

$$\sigma_s^2 = E_{\widetilde{\theta},\widetilde{K}^n}\left[\left(\frac{\widetilde{\theta}R\left(\widetilde{K}^n\right) - \left(\widetilde{K}^n - dK\right) + W\left(\widetilde{\theta}, V\left(\widetilde{\theta}, \widetilde{K}^n\right), \widetilde{K}^n\right) - W}{W} - \overline{r}\right)^2\right].$$

Let

$$Z = E_{\widetilde{\theta},\widetilde{K}^n}\left[\widetilde{\theta}R\left(\widetilde{K}^n\right) - \widetilde{K}^n + W\left(\widetilde{\theta}, V\left(\widetilde{\theta}, \widetilde{K}^n\right), \widetilde{K}^n\right)\right].$$

Then we can write

$$\overline{r} = \frac{Z + dK - W}{W}$$

and

$$\sigma_s^2 = E\left[\left(\frac{\widetilde{\theta}R\left(\widetilde{K}^n\right) - \widetilde{K}^n + W\left(\widetilde{\theta}, V\left(\widetilde{\theta}, \widetilde{K}^n\right), \widetilde{K}^n\right) - Z}{W}\right)^2\right]. \tag{41}$$

If $\underline{K}^n > dK$ then a slight increase ΔK in the level of capital does not change the optimal policy. Thus, the numerator in (41) remains constant, while the denominator increases to $W + d\Delta K$. The new variance is therefore

$$\sigma_{s+\Delta}^2 = E\left[\left(\frac{\widetilde{\theta}R\left(\widetilde{K}^n\right) - \widetilde{K}^n + W\left(\widetilde{\theta}, V\left(\widetilde{\theta}, \widetilde{K}^n\right), \widetilde{K}^n\right) - Z}{W + d\Delta K}\right)^2\right] < \sigma.$$

The proof for the case $\overline{K}_{s+\Delta}^n < dK$ is similar. ■

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